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AFFINELY ORDERED LIE GROUPS AND AXIOMATIZATION OF PSEUDO-EUCLIDEAN GEOMETRY

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This note is concerned with the affine orders on 3-dimensional solvable Lie groups, i.e., orders specified, in the affine coordinates belonging to an affine structure on a group, by families of cones. Our object is to determine the connection between the different affine structures on a 3-dimensional Lie group and the orders that are realized as families of cones relative to these structures. The result (Theorem 1) is then used to axiomatize pseudo-Euclidean geometry.

1

Let G be a 3-dimensional connected simply-connected solvable Lie group, and

$$\alpha_i\colon G \to \mathrm{Aff}(\mathbb{R}^3), \qquad i = 1, 2, \tag{1}$$

a simply transitive affine action of G on $\mathbb{R}^3$. Here $\mathrm{Aff}(\mathbb{R}^3)$ means the group of all affine transformations of a 3-dimensional arithmetical space. The actions α_1 and α_2 are called *affinely conjugate* if there exists an affine bijection $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\alpha_1(g) = A \circ \alpha_2(g) \circ A^{-1}$ for every $g \in G$.

By a *left-invariant affine structure* on G is meant a smooth structure for which all the transition functions and left translations, expressed in terms of coordinates, extend to transformations in $\mathrm{Aff}(\mathbb{R}^3)$. The simply transitive affine action (1) determines a *complete* left-invariant affine structure $\mathscr{A}_i$ on the group G. Indeed, consider the diffeomorphisms

$$\varphi_i\colon G \to \mathbb{R}^3, \qquad G \ni g \overset{\varphi_i}{\to} \alpha_i(g)(e) = x(i) = (x^1, x^2, x^3),$$

$i = 1, 2$, where $e \in \mathbb{R}^3$ is a fixed point. The numbers x^1, x^2, x^3 constitute an affine coordinate system on $\mathbb{R}^3$. They can therefore be used as affine coordinates $\mu_i\colon g \to (x^1, x^2, x^3)$ on G.

Two affine structures $\mathscr{A}_1$ and $\mathscr{A}_2$ on G are called *equivalent* if there exists an affine bijection $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the coordinate systems $G \overset{\mu_1}{\to} \mathbb{R}^3$ and $G \overset{\mu_2}{\to} \mathbb{R}^3$, belonging to $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively, satisfy the equality $\mu_2 \circ \mu_1^{-1} = A$.

Proposition. *Two left-invariant affine structures on a Lie group G, given by affine actions α_1, α_2, are equivalent if and only if the actions are affinely conjugate.*

Consider on the group G a left-invariant partial order $\leq$, determining the family of subsets $\mathfrak{P} = \{P_x\colon x \in G\}$, where $P_x = \{y \in G\colon x \leq y\}$. It induces [4] a partial order on $\mathbb{R}^3$, given by the family of subsets

$$\mathfrak{P}_i = \varphi_i(\mathfrak{P}) = \{\varphi_i(P_g)\colon P_g \in \mathfrak{P}\}.$$

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The order $\mathfrak{P}_i$ is $\alpha_i(G)$-invariant; i.e., putting $P_{ix(i)} = \varphi_i(P_g)$, where $x(i) = \varphi_i(g)$, we have

$$\alpha_i(h)(P_{ix(i)}) = P_{i\alpha_i(h)(x(i))}$$

for every $h \in G$.

Consider the diffeomorphism

$$f = \varphi_2 \circ \varphi_1^{-1} \colon \mathbb{R}^3 \to \mathbb{R}^3. \tag{2}$$

Clearly, $f(P_{1x(1)}) = P_{2f(x(1))}$ for every point $x(1) \in \mathbb{R}^3$.

A complete left-invariant affine structure $\mathscr{A}$, induced by an action α on G, is called *normal* if in it the left translations L_a, where a is an element of a maximal abelian subgroup, have the form of parallel displacements: $[L_a(x)]^k = x^k + a^k$, $k = 1, 2, 3$. The corresponding simply transitive action α is also called normal.

Theorem 1. *Suppose the action α_1 is normal, and $\mathfrak{P}_1$ and $\mathfrak{P}_2$ consist of elliptical cones. Then the actions α_1 and α_2 are affinely conjugate, the corresponding left-invariant affine structures $\mathscr{A}_1$ and $\mathscr{A}_2$ are affinely equivalent, and the mapping* (2) *is an affine bijection.*

2. Axiomatization of pseudo-Euclidean geometry

Our object in this section is to single out the 3-dimensional pseudo-Euclidean space ${}_1E^3$ of signature $\langle + - - \rangle$ from among all affinely ordered homogeneous affine Lorentz manifolds. Our approach is based on the idea of constructing a synthetic theory of Lorentz manifolds, as outlined in [1].

Notation. $\Pi(3)$ is the group of isometries of the space ${}_1E^3$. We shall call it the *Poincaré group.*

Taking into account the theory of homogeneous spaces and Lie groups, we replace the problem of axiomatizing a homogeneous Lorentz manifold by that of axiomatizing (on the basis of the idea of partial order) the Lorentz Lie group G_3, i.e., a connected simply-connected solvable Lie group, supplied with a left-invariant Lorentz metric.

Axiomatizing the group G_3 together with a complete left-invariant affine structure on it amounts to solving in succession the following two problems:

1) Provide an abstract group G with a complete left-invariant affine structure.

2) Define on G a Lorentz metric g without using the notion of smooth tensor field.

The first problem was solved by V. K. Ionin [2], [3]. A structure of simply-connected affine manifold on an abstract group G is a structure of the form $\langle G, \Gamma, \Phi, \Psi \rangle$, where Γ is the set of all affine transformations of the real line $\mathbb{R}$, while the sets $\Phi \subset G^{\mathbb{R}}$ and $\Psi \subset \mathbb{R}^G$ satisfy the following conditions:

(AI1) For every $\varphi \in \Phi$ and $\psi \in \Psi$, the composite $\psi \circ \varphi$ belongs to Γ.

(AI2) Φ is maximal; i.e., if $f \colon \mathbb{R} \to G$ but $f \notin \Phi$, then there exists a $\psi \in \Psi$ such that $\psi \circ f \notin \Gamma$.

(AI3) Ψ is maximal; i.e., if $f \colon G \to \mathbb{R}$ but $f \notin \Psi$, then there exists a $\varphi \in \Phi$ such that $f \circ \varphi \notin \Gamma$.

(AI4) If $x, y \in G$, then there exists a $\varphi \in \Phi$ such that $x, y \in \varphi(\mathbb{R})$.

(AI5) If $x, y \in G$, $x \neq y$, then there exists a $\psi \in \Psi$ such that $\psi(x) \neq \psi(y)$.

In these terms, an affine transformation $h \colon G \to G$ is defined as a mapping such that $\psi \circ h \circ \varphi \in \Gamma$ for all $\varphi \in \Phi$ and $\psi \in \Psi$.

The set $\{\varphi(\mathbb{R}) \colon \varphi \in \Phi_0\}$, where $\Phi_0 \subset \Phi$ is the subset of nonconstant mappings, is by definition the set of lines in G. A ray with origin $x \in G$ is a set $\varphi(\mathbb{R}_+)$, where $\varphi \in \Phi_0$, $\mathbb{R}_+ = \{t \in \mathbb{R} \colon t \geq 0\}$, and $\varphi(0) = x$.

The dimension $\dim G$ of the affine structure is defined in a natural fashion (see [2]).

An affine structure $\langle G, \Gamma, \Phi, \Psi\rangle$ is left-invariant if every left translation $L_a: x \to ax$ is an affine transformation.

A left-invariant n-dimensional affine structure $\langle G, \Gamma, \Phi, \Psi\rangle$ is called *normal* if for every $x, y \in G$ and $t \in T$, where T is a maximal abelian subgroup of G, there exists a unique $z \in G$ such that $\psi(z) - \psi(y) = \psi(L_t(x)) - \psi(x)$ for all $\psi \in \Psi$.

A point $a \in M \subset G$, where G is supplied with an affine structure, is called *interior for* M if for every $\varphi \in \Phi_0$ such that $a \in \varphi(\mathbb{R})$ there exist $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, such that $a \in \varphi((\alpha, \beta))$ and $\varphi((\alpha, \beta)) \subset M$. The set of interior points for M will be denoted by $\mathrm{int} M$.

The second problem is what we now solve, in the following way. Let G be an abstract group, supplied with a left-invariant affine structure $\langle G, \Gamma, \Phi, \Psi\rangle$. Suppose given on G a left-invariant order $\mathfrak{P} = \{P_x : x \in G\}$ satisfying the following conditions:

(AP1) The set $P \equiv P_e$, where e is the group identity, is a cone with vertex e, i.e., the union of rays with vertex e.

(AP2) The affine hull of the set P, i.e., the union of all the lines intersecting P in at least two points, coincides with G.

(AP3) The set P is closed; i.e., all points of the set $G \backslash P$ are interior.

(AP4) The cone P is elliptical; i.e., for any two points $x, y \in P$, $x, y \neq e$, $x \neq y$, that are not interior points for P, there exists an affine transformation $f \in \mathrm{Aff}(G)$ such that $f(e) = e$, $f(P) = P$, and $f(x) = y$.

(AP5) The group $\mathrm{Aut}(\mathfrak{P})_e$ acts transitively on $\mathrm{int}P$; here $\mathrm{Aut}(\mathfrak{P})_e$ means the stabilizer at e of the group $\mathrm{Aut}(\mathfrak{P})$ of all order automorphisms, i.e., all bijections $f: G \to G$ such that $f(P_x) = P_{f(x)}$.

Theorem 2. *Let G be an abstract group, supplied with a left-invariant normal affine 3-dimensional structure $\langle G, \Gamma, \Phi, \Psi\rangle$ and a left-invariant order $\mathfrak{P}$, satisfying conditions* (AP1)–(AP5). *Then G admits the structure of a connected simply-connected solvable Lie group with a left-invariant complete affine structure induced by the Ionin structure $\langle G, \Gamma, \Phi, \Psi\rangle$ and a left-invariant flat Lorentz metric g such that, in some global affine coordinates x_1, x_2, x_3,*

$$(3)\quad P_x = \left\{ (z_1, z_2, z_3) \in \mathbb{R}^3 : \sum_{\substack{i,j=1 \\ x=(x_1,x_2,x_3)}}^{3} g_{ij}(z_i - x_i)(z_j - x_j) \geq 0 \text{ and } z_i \geq x_i \right\},$$

$$(4)\qquad g_x(\xi, \eta) = g_{ij} \cdot \xi^i \eta^j, \qquad g_{ij} = \text{const.}$$

Consequently, the order $\mathfrak{P}$ is causal with respect to the metric g, i.e., any vector ξ issuing from a point x and lying in the cone P_x is nonspacelike at x with respect to g, and G is a simply transitive subgroup of the Poincaré group $\Pi(3)$.

Thus, Theorem 2 singles out those groups that on the one hand admit a pseudo-Euclidean geometry, but on the other hand are such that a metric for this geometry in a global affine chart must induce a relativistic causal order (3), (4). The group, affine, and order structures on G that are linked together by the axioms (AI1)–(AI5) and (AP1)–(AP5) determine, manifestly, a pseudo-Euclidean geometry on G with no requirement that G be abelian. If G is a priori *not* abelian, then the abelian subgroup necessarily acting on G in a simply transitive fashion (and therefore also carrying a pseudo-Euclidean structure) can be extracted from the group $\mathrm{Aut}(\mathfrak{P})$, which coincides with the semidirect product $\Pi(3) \times \{\text{homotheties}\}$.

We see, therefore, that a pseudo-Euclidean geometry on a group is not as strongly connected with the group's being abelian as was tacitly supposed in previous studies in the axiomatic theory of relatively (see the survey [4]).

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