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The Nash equilibrium of forest ecosystems

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Abstract: To find the possible equilibrium states of forest ecosystems one are suggested to use the theory of differential games. At within the 4-tier model of mosaic forest communities it established the existence of the Nash equilibrium states in such ecosystems.

Key–Words: Forest ecosystem, the equilibrium of the ecosystem, differential game, Nash equilibrium

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1 Introduction

As a rule, the *stationary equilibrium state* of the system, or *stationary equilibrium*, is *stationary state* for which characterizing its parameter $x(t)$ does not change with time, i. e.

$$\frac{dx}{dt} = 0.$$

However, the systems are often controlled by external factors $u_1, ..., u_N$, and in fact their dynamics is described by the differential equation of the form

$$\frac{dx}{dt} = f(t, x, u_1, ..., u_N).$$

In this case, it can consider this equation in the framework of optimal control theory, and moreover, in the framework of the theory differential games, and to find the so-called the *Nash equilibrium*.

In the theory of differential games each controlling factor u_i is considered to be in possession of the *player* i who tries to use it to affect the system so to have a maximal winning or minimal losing. Player's wining/lossing is described some given function $J_i(x, u_1, ..., u_N)$. Clearly, in reality, it is difficult to suggests that the factors can be changed completely independently from each other, and therefore, in the system can be installed in some sense of equilibrium.

Nash equilibrium in this case means that if each player is trying to unilaterally change their management strategy in the while other players policy remains unchanged, it has the worst record (greater loss).

Forest ecosystem dynamics can also be described by the differential equation with external control factors. As external controlling factors may be considered such characteristics of forest communities as a mosaic state m, interspecific and intraspecific competition k, the impact of the anthropogenic a and soil moisture w.

It is natural to try to establish the existence of Nash equilibrium in forest ecosystems with external control factors k, m, a, w.

2 Model of 4-tier mosaic forest

In [1, 2] was offered the next model 4-tier mosaic forest communities, characterized by productivity x:

$$\frac{dx}{dt} = -\frac{\partial}{\partial x} V(x, k, m, a, w), \qquad (1)$$

where

$$V(x, k, m, a, w) =$$

$$= \frac{\alpha}{6} x^6 + kx^4 + mx^3 + ax^2 + wx, \qquad (2)$$

$\alpha = \alpha_1\alpha_2\alpha_3\alpha_4 = const > 0$ are tiers of forest.

In [2] a stationary equilibriums of this ecosystem is completely studied in detail.

Below we examine Nash equilibriums and install them the existence for a 4-tier mosaic forest ecosystem.

3 Stationary equilibriums of trier mosaic forest

Stationary equilibriums $x = x(k, m, a, m)$ of trier mosaic forest we find by solving the equation

$$\frac{\partial}{\partial x} V(x, k, m, a, w) = 0. \qquad (3)$$

Consider the set

$$M_V = \{(x, k, m, a, w) : \frac{\partial}{\partial x} V =$$

$$= 6x^5 + 4kx^3 + 3mx^2 + 2ax + w = 0\},$$

which is contains of maximums, minimums and points of inflection of function $V_{(k,m,a,w)}(x) = V(x, k, m, a, w)$. All these poins are stationary equilibriums of given forest ecosystem.

We can change the factors (k, m, a, w) and to get different stationary equilibriums. In some cases the transition from one equilibrium to another is jump $x(k, m, a, w) \to (k', m', a', w')$, which is called *butterfly catastrophe*.

The behavior of the forest ecosystem in such catastrophes is investigated in [2].

We shall study behavior of the forest ecosystem under the Nash equilibriums.

4 The algorithm for finding Nash equilibriums

It is natural to consider the differential game with zero sum, because "winnings" of our players are poorly connected.

If a player forms its control action in the form of only time function $u(t)$ for the whole duration of the game, then $u(t)$ is *program control*. However, the player may select its control depending on the position x at time t of system. In this case, the player constructs a control action as a function of $u(t, x)$, which already dependent on the position $\{t, x\}$, and for $u(t, x)$ is used the term *positional control* [3]. Often we simply write $u(x)$.

We will look for positional control, positional Nash equilibrium.

For differential game with N players

$$\frac{dx}{dt} = f(x) + \sum_{j=1}^{N} g_j(x)u_j, \quad f(0) = 0,$$

$$x \in \mathbb{R}, \quad u_j \in \mathbb{R},$$

$$J_i(x, u_1, ..., u_N) = \int_0^{+\infty} [Q_i(x) + \sum_{j=1}^{N} R_{ij}(u_j)^2] dt,$$

$$(i = 1, ..., N),$$

$$Q_i > 0, \quad R_{ii} > 0, \quad R_{ij} \geq 0,$$

the existence problem of Nash equilibrium

$$J_i(u_1^*, u_2^*, u_i^*, ..., u_N^*) \leq$$

$$J_1(u_1^*, u_2^*, ..., u_{i-1}^*, u_i, u_{i+1}^*, ..., u_N^*), \qquad (4)$$

$$\forall u_i, \quad i = 1, ..., N,$$

is reduced to extremely complex the problem of finding positive definite solutions $V_i(x) > 0$ of nonlinear the Hamilton-Jacobi equations

$$(V_i)'_x(x) f(x) + Q_i(x) -$$

$$-\frac{1}{2}(V_i)'_x \sum_{j=1}^{N} [G_j(x)]^2 (R_{jj})^{-1} (V_j)'_x +$$

$$+\frac{1}{4}\sum_{j=1}^{N} R_{ij}[g_j(x)]^2 [(R_{jj})^{-1}]^2 [(V_j)'_x]^2 = 0, \qquad (5)$$

Then positional Nash equilibriums are

$$u_i^*(x) = u_i(V_i(x)) = -\frac{1}{2} R_{ii}^{-1} g_i(x) (V_i)'_x. \qquad (6)$$

$$(i = 1, ..., N).$$

(see [4, Theorem 10.4-2]).

5 Nash equilibrium of the forest ecosystem

In our case $N = 4$, player 1 is competition of trees $u_1 = k$, player 2 is mosaic factor $u_2 = m$, player 3 is anthropogenic interference $u_3 = a$ in the forest ecosystem (deforestation, fires, and so on.), and, finally, the player 3 is soil moisture $u_4 = w$.

Further

$$f(x) = -\alpha x^5, \quad g_1(x) = -4x^3,$$

$$g_2(x) = -3x^2 \quad g_3(x) = -2x, \quad g_4(x) = -1$$

and we take

$$R_{11} = R_{22} = R_{33} = R_{44} = 1, R_{ij} = 0 \, (i \neq j).$$

The Hamilton-Jacobi equations are:

$$Q_i + (V_i)'_x f(x) -$$

$$-\frac{1}{2}(V_i)'_x F(x) + \frac{1}{4}[g_i(x)]^2[(V_i)'_x]^2 = 0 \quad (7)$$

$$(i = 1, 2, 3, 4),$$

where

$$F(x) = \sum_{j=1}^{4} [g_i(x)]^2 (V_i)'_x.$$

Assuming that

$$V_1(x) = V_2(x) = V_3(x) = V_4(x) = \frac{1}{2}x^2 > 0,$$

we obtain the Hamilton-Jacobi equation in the form

$$Q_1 = \alpha x^6 + 4x^8 + \tfrac{9}{2}x^6 + 2x^4 + \tfrac{1}{2}x^2,$$
$$Q_2 = \alpha x^6 + 8x^8 + \tfrac{4}{9}x^6 + 2x^4 + \tfrac{1}{2}x^2,$$
$$Q_3 = \alpha x^6 + 8x^8 + \tfrac{9}{2}x^6 + x^4 + \tfrac{1}{2}x^2,$$
$$Q_4 = \alpha x^6 + 8x^8 + \tfrac{9}{2}x^6 + 2x^4 + \tfrac{1}{4}x^2.$$

Since all functions Q_i are positive definite, then the Hamilton-Jacobi equations are held for those features and for functions V_i that were selected above.

Therefore by Theorem 10.4-2 of [4] we have a Nash equilibrium

$$k^* = 2x^4, \quad m^* = \frac{3}{2}x^3, \qquad (8)$$
$$a^* = x^2, \quad w^* = \frac{1}{2}x,$$

found by the formulas (6).

We have the following winning / losing functions:

$$J_1(x, k, m, a, w) = \int_0^{+\infty} [Q_1(x) + k^2]dt,$$

$$J_2(x, k, m, a, w) = \int_0^{+\infty} [Q_2(x) + m^2]dt,$$

$$J_3(x, k, m, a, w) = \int_0^{+\infty} [Q_3(x) + a^2]dt,$$

$$J_4(x, k, m, a, w) = \int_0^{+\infty} [Q_4(x) + w^2]dt,$$

Productivity x under the Nash equilibrium (8) is found by integration of equation (1)–(2) and is satisfies the equatution

$$\int \frac{x^{-1}dx}{8x^6 + (9/2 + \alpha)x^4 + 2x^2 + 1/2} =$$

$$= -t + C, \qquad (9)$$

where C is constant of integration, or

$$2\ln(x) -$$

$$-\sum_R \frac{(16R^2 + 9R + 2Ra + 4)\ln(x^2 - R)}{(48R^2 + 18R + 4Ra + 4)} =$$

$$= -t + C,$$

where

R is root of $16Z^3 + (9 + 2a)Z^2 + 4Z + 1 = 0$.

For $\alpha = 0,0007$, i. e. for forest with 70% of upper tier mass of and for 10% in third others we have the following solution:

$$2.0\ln(x) - 0.772183\ln(x^2 + 0.354611) -$$

$$-0.113908\ln((x^2 + 0.103988)^2 + 0.165435) +$$

$$+0.731466\arctan(\frac{0.406737}{x^2 + 0.103988}) =$$

$$= -t + C.$$

In the case $C = -100$ this solution is presented at fig. 1.

We see that over time productivity of phytocenosis falls. In other words, the forest is degraded. But degradation comes no sooner than 100 years. We have a climax forest.

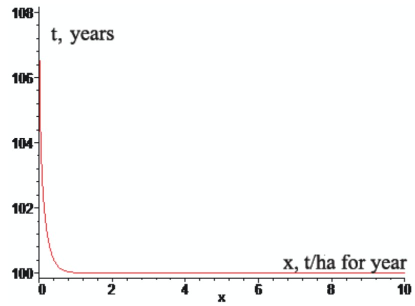

Figure 1: Forest productivity in the condition of the Nash equilibrium (8).

6 Conclusion

We have shown that it is possible to apply the theory of differential games to the study of forest ecosystems. We have shown that in such ecosystem there exist the Nash equilibriums that is installed in the system when reached some defined mediated the connection between the external factors affecting on the productivity of the forest.

As further research is necessary and useful to determine which forests and in some cases are in Nash equilibrium, and how it is expressed in terms of the traditional science on forests and forest ecosystems.

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