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A REMARK ON THE HELMHOLTZ-LIE PROBLEM

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In this note an answer is given to the following question: In what case is the universal covering space of a metric space M with metric ρ isometric to the Euclidean plane, the Lobachevsky plane, or the two-sphere, when M is assumed to permit rotation in Busemann's sense on some open ball neighborhood $B(x, \delta_x)$, $\delta_x > 0$, of each of its points (i.e., for any points a, a', b and b' in $B(x, \delta_x)$ such that $\rho(xa) = \rho(xa')$, $\rho(xb) = \rho(xb')$, and $\rho(ab) = \rho(a'b')$, there exists an isometry of the ball $B(x, \delta_x)$ onto itself, keeping x fixed and sending a to a' and b to b')?

This problem can be regarded as the local version of the famous Helmholtz-Lie problem. The most satisfactory solution of this problem is given in Freudenthal's paper [2]. He obtained the following result. Let M be a connected locally compact metric space and let F be a doubly transitive (i.e. sending any given pair of points onto any other) group of homeomorphisms of M, satisfying the following axioms:

(S) For any closed subsets A and B such that $A \cap B = \varnothing$, there exists a nonempty open subset U such that for any $\lambda \in F$, either $\lambda(U) \cap A = \varnothing$ or $\lambda(U) \cap B = \varnothing$.

(V) F is a complete group.

(Z) Let J_{x_0} be the stationary subgroup of F for the point $x_0 \in M$. There exists an orbit $J_{x_0}(y)$ separating the space M.

Then M is a doubly transitive homogeneous space in the sense of Birkhoff and Wang (in particular, Euclidean, hyperbolic or spherical space), and F is a closed subgroup of the isometry group of the corresponding space.

Analogous results have been obtained by other investigators [3]–[6]. Methods working globally and relying mainly on the results of Lie group theory are characteristic of these papers. On the other hand, we also have the following local solution of the Helmholtz-Lie problem, due to Busemann.

THEOREM B. *If each point x of a G-space $\langle M, \rho\rangle$ has a spherical neighborhood $B(x, \delta_x)$, $\delta_x > 0$, permitting rotation in Busemann's sense, then the universal covering space of M is elementary, i.e. Euclidean, hyperbolic, or spherical* ([1], §52).

Let us recall that by a *G-space* Busemann means a space satisfying the following axioms:

I. $\langle M, \rho\rangle$ is a metric space.

II. $\langle M, \rho\rangle$ is boundedly compact, i.e. every bounded infinite subset has at least one point of accumulation.

III. For any two distinct points x and z there exists a point y such that (xyz), i.e. y is distinct from x and z, and $\rho(xy) + \rho(yz) = \rho(xz)$.

IV. For each point x there is a positive number ρ_x such that for any two distinct points y and z in the open ball $B(x, \rho_x)$ there exists a point u for which (yzu).

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V. If (xyz_1) and (xyz_2) and $\rho(yz_1) = \rho(yz_2)$, then $z_1 = z_2$.

Although Busemann's result is the only attempt known to us to solve the Helmholtz-Lie problem from a local point of view, one cannot deduce from it any definitive answer to the question posed at the beginning of the present paper. The problem is that the axioms IV and V directly imply the local uniqueness of shortest arcs joining points $x, y \in B(p, \rho_p)$, and this is a very strong hypothesis, which should be avoided. We remark that introducing the concept of G-spaces raises more general questions, besides the immediate solution of the Helmholtz-Lie problem. We state below a set of axioms whose aim is to give an answer to the question formulated at the beginning of this paper.

We consider below only separable locally compact metric spaces M with intrinsic metric ρ.

Denote by $r(x)$ the least upper bound of all numbers $r > 0$ such that the sphere $S(x, r) = \{y \in M: \rho(xy) = r\}$ is a compact set. Furthermore, let $p(x)$ be the least upper bound of all numbers p_x corresponding to the point $x \in M$ and such that if $y, z \in B(x, p_x)$, then y is connected to z by a shortest arc. It is known that $p(x)$ either takes the value $+\infty$ for all x, or else it is everywhere finite and continuous.

Let us formulate the following two axioms:

(A_1) To each point $x \in M$ there corresponds a number $d(x) > 0$ having the property that, if $I(x)$ denotes the group of all self-isometries of the ball $B(x, d(x))$, then $I(x)$ acts effectively and transitively on each sphere $S(x, r)$, where $0 < r < d(x)$ and $\lambda(x) = x$ for $\lambda \in I(x)$.

(A_2) For each point $x \in M$ there exists a number $\delta_x > 0$ such that $\delta_x < \min(d(x), r(x), p(x))$, and satisfying the following conditions:

a) The sphere $S(x, r)$ is connected for each r, $0 < r \leqslant \delta_x$.

b) There exist two distinct points a_r, b_r on each sphere $S(x, r)$, $0 < r \leqslant \delta_x$, which separate $S(x, r)$, i.e., $S(x, r)\setminus\{a_r, b_r\} = A_1 \cup A_2$, where $A_1 \cap A_2 = \varnothing$, and A_1 and A_2 are nonempty and open in $S(x, r)\setminus\{a_r, b_r\}$.

c) For each r, $0 < r \leqslant \delta_x$, there exists an isometry $\lambda \in I(x)$ such that $\lambda(a_r) \in A_1$ and $\lambda(b_r) \in A_2$ (or vice versa).

Note that axiom (A_2) is a local version of Freudenthal's axioms (S) and (Z). Moreover, condition b) has the purpose of fixing the dimension of spheres, keeping in mind one-dimensional spheres and, consequently, two-dimensional space.

THEOREM 1. *Let the space M satisfy axioms (A_1) and (A_2). Then the following assertions are true*:

1) *$S(x, r)$, $0 < r \leqslant \delta_x$, is homeomorphic to the one-dimensional Euclidean sphere S^1.*

2) *The group $I(x)$ is a compact one-dimensional Lie group, each stationary subgroup being compact and zero-dimensional.*

3) *The connected component of the identity in $I(x)$ acts effectively and transitively on $S(x, r)$, $0 < r \leqslant \delta_x$, and is isomorphic to the Lie group $SO(2)$.*

The proof of Theorem 1 is based on the theory of the order of a topological space at a point [8]. We show that $\mathrm{ord}_p S(x, r) = 2$, and therefore by Theorem 8″ in [8], §51.VI, we conclude that $S(x, r)$ is homeomorphic to S^1. The rest follows from Lie group theory.

THEOREM 2. *If a space M satisfies axioms (A_1) and (A_2), then M is a two-dimensional topological manifold.*

For the solution of our problem, we need three more axioms.

(A_3) For each point $x \in M$, the ball $B(x, d(x))$ permits rotation in Busemann's sense.

(A_4) For each point $x \in M$, there is a point $y \in M$ such that $\rho(xy) < \min(d(x), d(y), \delta_y)$.

(A_5) Between any two shortest arcs emerging from any point $x \in M$ there exists an angle in A. D. Aleksandrov's sense, and moreover there exist at least two shortest arcs emerging from x, having a nonzero angle between them.

Axioms (A_3) and (A_4) are simply strengthenings of axiom (A_1); (A_4) allows us to rule out spaces with polyhedral metrics, which are not elementary.

DEFINITION. A space M which satisfies axioms (A_1)–(A_5) is called an *r-space.*

THEOREM 3. *Let M be an r-space. Then for each point $x \in M$ one can find a positive number η_x, $0 < \eta_x \leqslant \delta_x$, such that every point $y \in B(x, \eta_x)$ is joined to x by a unique shortest arc.*

This theorem is the key in all our investigations. Its proof is very complicated and takes a lot of space.

The following theorem answers the question which is resolved in this note.

THEOREM 4. *Let M be a complete r-space. Then M is a two-dimensional Busemann G-space whose universal covering space is elementary, i.e. it is the Euclidean plane, the Lobachevsky plane, or the two-sphere.*

COROLLARY. *A complete r-space must be one of the following spaces: two-sphere, projective plane, Euclidean plane, cylinder, torus, Möbius band, Klein bottle, or, finally, a locally-hyperbolic two-dimensional space, of which there are an infinite number, but they can be described in a well-known way* ([7], *Chapter* IX, §2B).

REMARK. An *r*-space which is not complete is not a Busemann *G*-space. Moreover, looking at a punctured Euclidean plane shows that there exist noncomplete *r*-spaces whose universal covering space is not isometric to an elementary space.

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