UNIQUENESS OF ABELIAN AFFINE CHRONOGEOMETRY

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In the axiomatic construction of the pseudo-Euclidean geometry of signature $(+\ -\ \dots\ -)$, or the geometry of space-time of the special theory of relativity, based on the concept of partial order, it is generally assumed that an order is given on some topological space V, with an Abelian group G of transformations whose action on V is simply transitive and leaves the order in question invariant. It is proved that the space V is affine, the order is determined by a family of cones, and the group G is represented as a group of parallel translations (see [1]). Thereafter, the space V is furnished with the pseudo-Euclidean structure with the help of the obtained family of equal parallel cones. However, the question of uniqueness of the result is cast aside. As a matter of fact, an Abelian group can have distinct affine-nonequivalent simply transitive representations in $\mathbb{R}^n$ [2], and, consequently, there arises the question of existence of affine-nonequivalent causal theories for space-time of the special theory of relativity, or chronogeometries. The aim of the present article is to prove uniqueness for the Abelian affine chronogeometry. In other words, the conventional affine interpretation of the special theory of relativity presented in most textbooks and monographs is unique.

Let G be an Abelian connected simply-connected Lie group of dimension $n \geq 3$ and let

$$\alpha_i : G \to \operatorname{Aff}(\mathbb{R}^n) \quad (i = 1, 2)$$

be a simply transitive affine action of G on $\mathbb{R}^n$. Here $\operatorname{Aff}(\mathbb{R}^n)$ stands for the group of all affine transformations of the n-dimensional arithmetic space. The actions α_1 and α_2 are called *affine-equivalent* if there exists an affine bijection $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $\alpha_1(g) = A \circ \alpha_2(g) \circ A^{-1}$ for every $g \in G$.

A left-invariant affine structure on G is a smooth structure in which all change-of-coordinate functions and left translations, being written in coordinates, are extendible to transformations of $\operatorname{Aff}(\mathbb{R}^n)$.

The simply transitive affine action $\alpha_i : G \to \operatorname{Aff}(\mathbb{R}^n)$ determines some left-invariant affine structure $\mathcal{A}_i$ on the Lie group G. Indeed, consider the diffeomorphisms

$$\varphi_i : G \to \mathbb{R}^n \quad (i = 1, 2),$$

$$G \ni g \xrightarrow{\varphi_i} \alpha_i(g)(e) = x(i) \doteq (x^1, \dots, x^n) \quad (i = 1, 2) \tag{1}$$

where $e \in \mathbb{R}^n$ is a fixed point. Here $x^1, \dots, x^n$ is an affine coordinate system in $\mathbb{R}^n$. These coordinates can be used as coordinates in G:

$$\begin{tikzcd}
G \ni g \arrow[d, "\varphi_i"] \arrow[drr, "\mu_i"] & & \\
\alpha_i(g)(e) = x(i) & \stackrel{\varepsilon}{=} & (x^1, \dots, x^n),
\end{tikzcd} \tag{2}$$

where ε is the mapping associating a vector $x \in \mathbb{R}^n$ with the tuple $(x^1, \dots, x^n)$. In these coordinates, each left translation $L_h : G \to G$ is determined by the affine bijection

$$\begin{aligned}
(\mu_i \circ L_h \circ \mu_i^{-1})(x^1, \dots, x^n) &= \big((\varepsilon \circ \varphi_i) \circ L_h\big)\big(\varphi_i^{-1} \circ \varepsilon^{-1}\big)(x^1, \dots, x^n) \\
&= (\varepsilon \circ \varphi_i)(L_h(g)) = \varepsilon(\alpha_i(h)(\alpha_i(g)(e))) = \varepsilon(\alpha_i(h)(\varepsilon^{-1}(\varepsilon(x(i))))) \\
&= (\varepsilon \circ \alpha_i(h) \circ \varepsilon^{-1})(x^1, \dots, x^n) \in \operatorname{Aff}(\mathbb{R}^n),
\end{aligned}$$

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which means in fact that the assigned affine structure on G is left-invariant.

We call two affine structures $\mathcal{A}_1$ and $\mathcal{A}_2$ on G *equivalent* if there exists an affine bijection $A : \mathbb{R}^n \to \mathbb{R}^n$ such that the equality $\mu_2 \circ \mu_1^{-1} = A$ holds for the coordinate systems

$$G \xrightarrow{\mu_1} \mathbb{R}^n, \quad G \xrightarrow{\mu_2} \mathbb{R}^n$$

that correspond to the structures $\mathcal{A}_1$ and $\mathcal{A}_2$.

Proposition. *The left-invariant affine structures on the Lie group G determined by affine actions α_1 and α_2 are equivalent if and only if the actions are affine-conjugate.*

PROOF. Let left-invariant affine structures $\mathcal{A}_1$ and $\mathcal{A}_2$ on the Lie group G be determined by affine actions α_1 and α_2 respectively. Assume $\mathcal{A}_1$ to be equivalent to $\mathcal{A}_2$; i.e., if μ_1 and μ_2 are the respective coordinate systems then $\mu_2 \circ \mu_1^{-1} = A$, where $A : \mathbb{R}^n \to \mathbb{R}^n$ is an affine bijection. We have relations (2) and

$$\varepsilon(x(2)) = (\mu_2 \circ \mu_1^{-1})(\varepsilon(x(1))) = A(\varepsilon(x(1))). \tag{3}$$

When g equals the unity of the group G, equality (3) takes the form

$$\varepsilon(e) = A(\varepsilon(e)). \tag{4}$$

It follows from (3) that

$$\varepsilon(\alpha_2(g)(e)) = (A \circ \varepsilon)(\alpha_1(g)(e)),$$

i.e., in view of (4),

$$(\alpha_2(g) \circ (\varepsilon^{-1} \circ A \circ \varepsilon))(e) = ((\varepsilon^{-1} \circ A \circ \varepsilon) \circ \alpha_1(g))(e). \tag{5}$$

Substituting the product hg for g in (5), we derive

$$\alpha_2(h)(\alpha_2(g)(e)) = ((\varepsilon^{-1} \circ A \circ \varepsilon) \circ \alpha_1(h))(x(1)),$$

and, in view of (3),

$$\alpha_2(h)((\varepsilon^{-1} \circ A \circ \varepsilon)(x(1))) = ((\varepsilon^{-1} \circ A \circ \varepsilon) \circ \alpha_1(h))(x(1))$$

or

$$\alpha_2(h) \circ (\varepsilon^{-1} \circ A \circ \varepsilon) = (\varepsilon^{-1} \circ A \circ \varepsilon) \circ \alpha_1(h).$$

Consequently,

$$\alpha_1(g) = (\varepsilon^{-1} \circ A \circ \varepsilon)^{-1} \circ \alpha_2(g) \circ (\varepsilon^{-1} \circ A \circ \varepsilon).$$

Assigning $\widehat{A} = \varepsilon^{-1} \circ A \circ \varepsilon$, we obtain

$$\alpha_1(g) = \widehat{A}^{-1} \circ \alpha_2(g) \circ \widehat{A}; \tag{6}$$

i.e., α_1 and α_2 are affine-conjugate.

Conversely, if α_1 and α_2 are affine-conjugate (i.e., equality (6) holds), then

$$\begin{aligned} x(2) = \alpha_2(g)(e) &= [\widehat{A} \circ \alpha_1(g) \circ \widehat{A}^{-1}](e) \\ &= [\widehat{A} \circ \alpha_1(g) \circ \widehat{A}^{-1}](\widehat{A}(e)) = \widehat{A}(\alpha_1(g)(e)) = \widehat{A}(x(1)) \end{aligned}$$

and, consequently,

$$\mu_2(g) = A(\mu_1(g)).$$

So, $\mu_2 \circ \mu_1^{-1} = A$, which means that the affine structures $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent. The proposition is proved.

Consider the left-invariant partial order on the Lie group G determined by the family of subsets $\mathfrak{P} = \{P_x : x \in G\}$ (see [1]). We have the partial order

$$\mathfrak{P}_i = \varphi_i(\mathfrak{P}) = \{\varphi_i(P_g) : P_g \in \mathfrak{P}\}$$

on $\mathbb{R}^n$ which is $\alpha_i(G)$-invariant; i.e., if we put $P_{ix(i)} = \varphi_i(P_g)$, where $x(i) = \varphi_i(g)$, then

$$\alpha_i(h)(P_{ix(i)}) = P_{i\alpha_i(h)(x(i))}$$

for all $h \in G$.

Look at the diffeomorphism

$$f = \varphi_2 \circ \varphi_1^{-1} : \mathbb{R}^n \to \mathbb{R}^n. \tag{7}$$

It is obvious that

$$f(P_{1x(1)}) = P_{2f(x(1))} \tag{8}$$

for all points $x(1) \in \mathbb{R}^n$.

Theorem. *Let the action α_1 be the group of parallel translations on $\mathbb{R}^n$, and let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ consist of strictly convex cones. Then the actions α_1 and α_2 are affine-conjugate and the corresponding left-invariant affine structures $\mathcal{A}_1$ and $\mathcal{A}_2$ are affine-equivalent.*

PROOF. It follows from (7), (8), and also from [3] that f is an affine bijection and, consequently, in coordinates (2) we have

$$\begin{aligned} \mu_2 \circ \mu_1^{-1} &= (\varepsilon \circ \varphi_2) \circ (\varepsilon \circ \varphi_1)^{-1} \\ &= \varepsilon \circ (\varphi_2 \circ \varphi_1^{-1}) \circ \varepsilon^{-1} = \varepsilon \circ f \circ \varepsilon^{-1} \in \operatorname{Aff}(\mathbb{R}^n). \end{aligned}$$

This means that the affine structures $\mathcal{A}_1$ and $\mathcal{A}_2$ are affine-equivalent. It follows from the proposition that the actions α_1 and α_2 are conjugate. The theorem is proved.

References

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