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A TOPOS-THEORETIC APPROACH TO THE FOUNDATIONS OF RELATIVITY THEORY

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In most cases, axiomatic descriptions of relativity theory use set theory. And although this has led to certain successes (see [1] and [2]), it should be noted that the existing sets of axioms for special relativity do not allow one to declare with satisfaction that the construction of this fundamental physical theory has been completed. The situation of the axiomatic approach to general relativity is still more complicated [2].

From a geometric point of view it is preferable to speak about constructing a unified synthetic theory of pseudo-Euclidean and pseudo-Riemannian manifolds. However, such a unified approach to the axiomatization of both special and general theories of relativity has apparently not been used. As a rule, the systems of axioms for special relativity contain fewer primary notions and relations, are simpler, and lead directly to the ultimate goal. In the case of general relativity it is difficult to introduce a pseudo-Riemannian metric, especially a smooth one, since this requires first solving the problem of endowing the set being considered (the space-time) with a structure of a (smooth) manifold [2].

Such difficulties, especially those of mathematical character, could be accounted for by the fact that we are dealing with the problem of describing two substantially different physical theories. In one case we deal with a mathematical theory of (flat) space-time which forms a background on which all sorts of physical fields, including the gravitational one, act on an equal footing. In the other case the problem consists in an axiomatic description of only one physical field, the gravitational one. However, first, such views are far from being accepted by all physicists, and, second, it is quite natural to try to construct a unified theory using, if the need arises, new mathematical ideas.

In the author's opinion, this state of the problem is due to the fact that one is trying to solve it on the basis of the set-theoretic approach. This has been a traditional approach of twentieth-century mathematics, but in the present case, concerning a mathematical description of the real space-time, one must seek the root cause of failures in the deficiencies of the mathematical apparatus being used. It is naive to think that all attributes of the space-time form of existence of matter can be formulated completely in terms of set theory. This theory is only a historical product of consciousness; it came into being as a tool for analyzing infinity, but this tool is of limited use when analyzing space-time.

In this note we demonstrate the efficiency of a topos-theoretic approach to solving the problem of a unified axiomatic description of special and general theories of relativity. In other words, we present below a categorical theory of pseudo-Riemannian manifolds.

Let $\mathscr{E}$ be an elementary topos with continuous real numbers object $\mathbf{R}_T$ (see [3]).

An affine morphism $\alpha\colon \mathbf{R}_T \to \mathbf{R}_T$ is a finite composition of morphisms of the

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form

$$1_{\mathbf{R}_T}, \quad \oplus \circ (1_{\mathbf{R}_T} \times \mu) \circ j, \quad \otimes \circ (\lambda \times 1_{\mathbf{R}_T}) \circ j,$$

where $\oplus$ and $\otimes$ are the operations of addition and multiplication in $\mathbf{R}_T$, $\mu, \lambda: 1 \to \mathbf{R}_T$ are arbitrary elements of $\mathbf{R}_T$, and $j: \mathbf{R}_T \simeq 1 \times \mathbf{R}_T$ is an isomorphism. We denote by Γ the set of all affine morphisms from $\mathbf{R}_T$ to $\mathbf{R}_T$.

An affine object in $\mathscr{E}$ is an $\mathscr{E}$-object a together with two sets of morphisms Φ and Ψ:

$$\Phi \subseteq \mathrm{Hom}_{\mathscr{E}}(\mathbf{R}_T, a), \qquad \Psi \subseteq \mathrm{Hom}_{\mathscr{E}}(a, \mathbf{R}_T),$$

such that the following conditions hold:

1) For any $\varphi \in \Phi$, $\psi \in \Psi$, we have $\psi \circ \varphi \in \Gamma$.

2) If $f \in \mathrm{Hom}_{\mathscr{E}}(\mathbf{R}_T, a) \backslash \Phi$, then there is a $\psi \in \Psi$ such that $\psi \circ f \notin \Gamma$.

3) If $f \in \mathrm{Hom}_{\mathscr{E}}(a, \mathbf{R}_T) \backslash \Psi$, then there is a $\varphi \in \Phi$ such that $f \circ \varphi \notin \Gamma$.

4) For any monomorphisms $f: \Omega \rightarrowtail a$ and $g: \Omega \rightarrowtail \mathbf{R}_T$, there is a $\varphi \in \Phi$ such that $\varphi \circ g = f$.

5) For any monomorphisms $f: \Omega \rightarrowtail a$ and $g: \Omega \rightarrowtail \mathbf{R}_T$, there is a $\psi \in \Psi$ such that $\psi \circ f = g$.

Here Ω is the subobject classifier in $\mathscr{E}$.

In the category Set, affine objects are sets endowed with a structure of an affine space [4]. In the topos $\mathrm{Bn}(I)$ and in the spatial topos $\mathrm{Top}(I)$ (the notation being as in [5]), an affine object is a fiber bundle with base I and affine spaces as fibers.

Not every topos with a real numbers object has an affine object. For instance, this is the case for the topos $M_2 - \mathrm{Set}$.

A categorical description of special and general relativity presupposes that a Lorentz structure, i.e., a quadratic form g_{ij}, has been defined either in an affine space or in a fiber bundle with affine spaces as fibers. This can be done by defining in the affine space a family of equal and parallel elliptic cones [7].

Below, we shall use the definitions, concepts, and notation of [5].

Let a be an affine object in the topos $\mathscr{E}$. An order in a is an $\mathscr{E}$-object P together with a collection of subobjects $\{p_x: P \rightarrowtail a\}$, where $x: 1 \to a$ is an arbitrary element, such that 1) $x \in p_x$ and 2) $y \in p_x$ implies $z \in p_x$ for any $z \in p_y$.

In what follows, we shall use the notation $\mathscr{P} = \langle P, \{p_x\}\rangle$ for an order in a.

A morphism $f: a \to a$ is said to be *affine* if, for any $\varphi \in \Phi$ and $\psi \in \Psi$, we have $\psi \circ f \circ \varphi \in \Gamma$.

Denote by $\mathrm{Aff}(a)$ the set of all affine morphisms, and let $\mathscr{A} \subset \mathrm{Aff}(a)$ consist of commuting morphisms.

An order $\mathscr{P}$ is invariant under $\mathscr{A}$ if for every p_x, p_y there is a $g_{xy} \in \mathscr{A}$ such that $g_{xy} \circ p_x = p_y$. A morphism $f: a \to a$ preserves an order $\mathscr{P}$ if for each p_x there is a p_y such that $f \circ p_x = p_y$.

The collection of morphisms that preserve an order $\mathscr{P}$ invariant under $\mathscr{A}$ will be denoted by $\mathrm{Aut}(\mathscr{P})$.

A *ray* is a morphism $\lambda: \mathbf{R}^+ \rightarrowtail \mathbf{R}_T \xrightarrow{\varphi} a$, where $\varphi \in \Phi$. Here $\mathbf{R}^+$ is the subobject of $\mathbf{R}_T$ consisting of those t such that $0 \le t$ (see the definition of the order in $\mathbf{R}_T$ is [3]). An order $\mathscr{P}$ is said to be *conic* if 1) for each $y \in p_x$ there is a ray $\lambda \subset p_x$ such that $x, y \in \lambda$, and 2) x is the origin of λ, i.e., if λ' is a ray and $y \in \lambda' \subset \lambda$, $\lambda' \neq \lambda$, then $x \notin \lambda'$.

An order $\mathscr{P}$ *has an acute vertex* if for each p_x there is no $\varphi_x \in \Phi$ with $\varphi_x \subset p_x$. An order is *complete* if for each element $z: 1 \to a$ and each p_x there exist different elements $u_x, v_x: 1 \to a$ and $\varphi \in \Phi$ such that $z, u_x, v_x \in \varphi_x$ and $u_x, v_x \in p_x$.

An element $u \in p_x$ is said to be *extreme* if there is a $\varphi \in \Phi$ such that $u \in \varphi$ but $y \notin \varphi$ for all $y \in p_x$.

A conic order $\mathscr{P}$ is said to be *strict* if, for each nonextreme element $u \in p_x$, and $v \in p_x$, and each ray λ with origin u such that $v \in \lambda$, there exists an extreme element $w \in \lambda$, and $w \in p_x$.

An affine object a with an order $\mathscr{P}$, which is complete, strict, conic, has an acute vertex, and is invariant under $\mathscr{A}$, is said to be *Lorentz* if for each $x\colon 1 \to a$ and each extreme elements $u, v \in p_x$ with $u, v \neq x$ there is an $f \in \operatorname{Aut}(\mathscr{P})$ such that the diagrams

$$\begin{array}{ccc} a & \xrightarrow{f} & a \\ {\scriptstyle u}\nwarrow & & \nearrow{\scriptstyle v} \\ & 1 & \end{array} \qquad \begin{array}{ccc} a & \xrightarrow{f} & a \\ {\scriptstyle x}\nwarrow & & \nearrow{\scriptstyle x} \\ & 1 & \end{array}$$

commute.

THEOREM 1. *A Lorentz object in the category* Set *is an affine space admitting a pseudo-Euclidean structure defined by a quadratic form* $x^{02} - \sum_1^n x^{k2}$, *where* n *is finite or equal to* ∞, *and* $\operatorname{Aut}(\mathscr{P})$ *is the Poincaré group supplemented with similarities. A Lorentz object in* $\operatorname{Top}(I)$ *is a fiber bundle over* I *with fibers endowed with an affine structure and a continuous pseudo-Euclidean structure of finite or infinite dimension.*

Thus, the language of topos theory allows us to axiomatize in a unified way both special and general theories of relativity, the axioms being the same in both cases. Selecting one or another physical theory amounts to selecting a concrete topos. It is quite possible to take not only the topoi Set, $\operatorname{Bn}(I)$, or $\operatorname{Top}(I)$, but also any others that have an affine object. This means that there are essentially new generalizations of relativity theory.

Finally, let us note that, since the problem of characterizing the topos Set (more precisely, the problem of a categorical description of set theory), as well as the problem of characterizing the spatial topos $\operatorname{Top}(I)$ within the class of elementary topoi were solved long ago [6], we have in fact solved the problem of a categorical description of relativity theory.

THEOREM 2. *If* $\mathscr{E}$ *is a well-pointed topos satisfying the axiom of partial transitivity with a Lorentz object* a, *then* $\mathscr{E}$ *is a model of set theory* Z *and* a *is a model of special relativity, i.e., a model of a pseudo-Euclidean space of finite or infinite dimension. If* $\mathscr{E}$ *is a topos defined over* Set *that has enough points and satisfies the axiom* (SG) *(see* [6]*), then* $\mathscr{E}$ *is the topos* $\operatorname{Top}(I)$ *and* a *is a fiber bundle over* I *with fibers endowed with a pseudo-Euclidean structure.*

There is still another possibility of applying topos theory to a mathematical description of space-time. One can attempt to achieve the desired simplicity when axiomatizing relativity theory at the cost of giving up the classical view that space-time is the world of events "placed" in a single "space". To this end, consider a partially ordered set P and contravariant functors from the pre-order category P to the category Set. This gives rise to the topos Set^P, and it is this topos which is the new mathematical space-time. The value of a functor F on an element x of P is the set $F(x)$. The set P is interpreted as the collection of all possible situations of obtaining information about past. It has a (timelike) partial order. The set $F(x)$ is the (causal) past cone consisting of the events that are observed in situation x. The functor F can be interpreted as a time flow. The topos Set^P consists of all possible time flows. It is not hard to see that a classical Lorentz transformation corresponds to a natural isomorphism of functors, i.e., time flows.

Thus, the space-time $\mathbf{Set}^P$, which may be described as a Grothendieck topos, can no longer be "placed" in a single "space".

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