GRAVITATIONAL ANALOG OF THE ZEEMAN EFFECT

A. K. Guts

UDC 530.12:531.51

An equation is obtained for calculating the spin–gravitational interaction analogous to the Zeeman effect.

The purpose of this note is to obtain an equation for calculating the spin–gravitational interaction analogous to the Zeeman effect. In predicting this phenomenon, Zel'dovich [1] suggested that the effect is caused by the components $g_{0\alpha}$ ($\alpha = 1, 2, 3$) [2]. This is, however, not quite correct: even when these components are nonzero, the effect might not occur.

The equation we derive is invariant under an arbitrary transformation of coordinates.

In what follows, $i, k, l, \ldots = 0, 1, 2, 3$; $\alpha, \beta, \gamma, \ldots = 1, 2, 3$.

§1. Derivation of the Equation

Let $h^i_{(a)}$, $h_{(a)i}$ denote respectively the contravariant and covariant components of the tetrad distinguished by the index (a), so

$$h^{(a)\,i} = \eta^{(ab)}\,h^i_{(b)},\quad h^i_{(a)} = \eta_{(ab)}\,h^{(b)\,i},\quad g_{ik}\,h^i_{(0)}\,h^k_{(0)} > 0,$$

where $\eta_{(ab)} = \{1, -1, -1, -1\}$ is the Minkowski metric tensor. The connection between the tetrad field and the metric field is given by

$$g_{ik} = h_{(a)\,i}\,h^{(a)}_k,\quad g^{ik} = h^i_{(a)}\,h^{(a)\,k},\quad h^i_{(a)}\,h^{(a)}_k = \delta^i_k.$$

The tetrad field is defined to within the rotations

$$\tilde{h}_{(a)\,i} = \Omega^{(b)}_{(a)}(x)\,h_{(b)\,i},$$

where the matrix $\| \Omega^{(a)}_{(b)}(x) \|$ is orthogonal. The tetrads $h^{(a)}_i$ then transform by means of the orthogonal matrix

$$\tilde{\Omega}^{(a)}_{(b)}(x) = \eta^{(ac)}\,\Omega^{(f)}_{(c)}(x)\,\eta_{(fb)}. \tag{1}$$

We write the Dirac equation in the form

$$i\hbar\gamma^k\left(\frac{\partial\psi}{\partial x^k} - \Gamma_k\psi\right) + \frac{e}{c}\,\gamma^k A_k\,\psi - mc\psi = 0, \tag{2}$$

where

$$\Gamma_k = \frac{1}{4}\,g_{ml}\left(\frac{\partial h^{(s)}_r}{\partial x^k}\,h^l_{(s)} - \Gamma^l_{rk}\right)s^{mr} + a_k\cdot I$$

[3, p.381; 4, p. 131].

We now suppose that the required equation is of the form

Novosibirsk State University. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, Vol. 16, No. 9, pp. 30-33, September, 1973. Original article submitted June 19, 1972.

$$\mathfrak{Z} = \frac{\hbar c}{2}\sum_{\alpha} B^{\alpha}\sigma_{\alpha},$$

where the quantities B^{α} are real, do not contain Planck's constant $\hbar$, and are functions only of the tetrad components.

Equation (2) can be written as

$$\hat{D}_{\mp} u_{\pm} + \sum_{\tau} C_{\tau}\sigma_{\tau} u_{\mp} = 0,$$

where

$$\psi = \begin{pmatrix} u_{+} \\ u_{-} \end{pmatrix},$$

$$\hat{D}_{\mp} \equiv i\hbar h^{\kappa}_{(0)} \frac{\partial}{\partial x^{\kappa}} + \frac{\hbar}{2}\sum_{\alpha} B^{\alpha}\sigma_{\alpha} - i\hbar a_{\kappa} h^{\kappa}_{(0)} + \frac{e}{c} h^{\kappa}_{(0)} A_{\kappa},$$

$$B^{\alpha} \equiv A_{mr,\kappa}\left\{ h^{\kappa}_{(0)} \sum_{\gamma<\beta} h^{mr}_{(\gamma\beta)} \varepsilon_{\gamma\alpha\beta} + \sum_{\mu} h^{\kappa}_{(\mu)} \sum_{\gamma} h^{mr}_{(0\gamma)} \varepsilon_{\mu\gamma\alpha} \right\},$$

$$C_{\tau} \equiv i\hbar h^{\kappa}_{(\tau)} \frac{\partial}{\partial x^{\kappa}} - i\hbar P_{\tau} - h^{\kappa}_{(\tau)}\left(i\hbar a_{\kappa} - \frac{e}{c} A_{\kappa} \right),$$

$$P_{\tau} \equiv \frac{1}{2} A_{mr,\kappa}\left\{ h^{\kappa}_{(0)} h^{mr}_{(0\tau)} - \sum_{\mu} h^{\kappa}_{(\mu)} \sum_{\alpha<\beta} h^{mr}_{(\alpha\beta)} \sum_{\gamma} \varepsilon_{\alpha\gamma\beta}\varepsilon_{\mu\gamma\tau} \right\},$$

$$A_{mr,\kappa} = \frac{1}{2} g_{ml}\left(\frac{\partial h^{(s)}_{r}}{\partial x^{\kappa}} h^{l}_{(s)} - \Gamma^{l}_{r\kappa} \right),$$

$$h^{mr}_{(i\kappa)} \equiv h^{m}_{(i)} h^{r}_{(\kappa)} - h^{m}_{(\kappa)} h^{r}_{(i)},$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi–Civita symbol in the special theory of relativity.

We can write out the components B^{α} in detail:

$$B^{1} = A_{mr,\kappa}\{ - h^{\kappa}_{(0)} h^{mr}_{(23)} + h^{\kappa}_{(2)} h^{mr}_{(03)} - h^{\kappa}_{(3)} h^{mr}_{(02)} \},$$
$$B^{2} = A_{mr,\kappa}\{ h^{\kappa}_{(0)} h^{mr}_{(13)} - h^{\kappa}_{(1)} h^{mr}_{(03)} + h^{\kappa}_{(3)} h^{mr}_{(01)} \},$$
$$B^{3} = A_{mr,\kappa}\{ - h^{\kappa}_{(0)} h^{mr}_{(12)} + h^{\kappa}_{(1)} h^{mr}_{(02)} - h^{\kappa}_{(2)} h^{mr}_{(01)} \},$$

and we thus immediately get

$$B^{\alpha} = \frac{1}{2} A_{mr,\kappa} \sum_{i,p,l} \varepsilon^{(ipl\alpha)} h^{mr}_{(ip)} h^{\kappa}_{(l)}.$$

It is not difficult to see that only these quantities satisfy the above conditions. We have thus obtained the required equation. This equation has the following properties.

a) The quantity $\mathfrak{Z}$ does not vary under arbitrary transformations of coordinates because $A_{mr,k}$ and $h^{mr}_{(ip)}\, h^{k}_{(l)}$ are third-rank tensors;

b) As regards purely spatial rotations of tetrads with constant coefficients, i.e. when the matrix $\Omega^{(a)}_{(b)}$ has the form

$$\Omega^{(0)}_{(0)} = 1, \quad \Omega^{(\alpha)}_{(0)} = \Omega^{(0)}_{(\alpha)} = 0 \ (\alpha = 1, 2, 3),$$

the quantities B^{α} behave as pseudovectors. This means that $\mathfrak{Z}$ behaves like its analog in quantum field theory. This follows from (1).

The most interesting property is (a). It implies that the effect is independent of any arbitrariness in the choice of the coordinate system.

§ 2. Synchronous Frame of Reference

In a synchronous frame of reference the metric can be written as

$$ds^{2} = dx^{0^{2}} + g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}$$

[5]. It is then always possible to choose the tetrad as:

$$h_i^{(0)} = (1, 0, 0, 0), \quad h_i^{(\alpha)} = -h_{(\alpha)\,i} = (0, h_1^{(\alpha)}, h_2^{(\alpha)}, h_3^{(\alpha)}).$$

Evaluating, we get

$$B^\alpha = \frac{1}{2} \sum_{\gamma<\lambda} \varepsilon^{0\gamma\lambda\alpha} \frac{\partial g_{\beta\mu}}{\partial x^0} h^{\beta\mu}_{(\gamma\lambda)}$$
$$+ \sum_{\gamma<\beta} \sum_{\delta<\mu} \left(h_{(s)\,\delta} \frac{\partial h_\mu^{(s)}}{\partial x^0} - h_{(s)\,\mu} \frac{\partial h_\delta^{(s)}}{\partial x^0} \right) \varepsilon^{\gamma\beta 0\alpha} h^{\delta\mu}_{(\gamma\beta)}.$$

Noting that the first term is equal to zero, we obtain

$$B^\alpha = \frac{1}{2} h_{(s)\,[\delta} \frac{\partial h_{\mu]}^{(s)}}{\partial x^0} \varepsilon^{\gamma\beta 0\alpha} h^{\delta\mu}_{(\gamma\beta)}. \tag{3}$$

Since it can be assumed for a static field that the tetrad components do not depend on time, the effect does not occur in this case.

It is clear that the effect is absent if the metric can be converted by some coordinate transformation to diagonal form.

The effect does occur for a stationary field created by a rotating body [1; 4, pp. 144-147]. However, if we go over to a synchronous frame of reference then the effect must remain according to property (a) and this means that (3) can be nonzero.

The following conclusions can be drawn.

(i) The fact that the components $g_{0\alpha}$ are equal to zero does not imply that the effect is absent; on the other hand, even when the components are nonzero the effect can still be absent;

(ii) The presence or absence of the effect does not depend on whether or not the 3-vectors

$$\tilde{\Omega}_\alpha = -\frac{c}{2} \frac{\sqrt{-g}}{g_{00}} \varepsilon_{\alpha\beta\gamma} \left(\frac{1}{2} \frac{\partial g^{\beta\gamma}}{\partial x^0} + g^{\beta\mu} \Gamma^\gamma_{\mu 0} \right)$$

are equal to zero and

$$\Omega = -\frac{c}{2} \operatorname{curl} \overrightarrow{(g_{01}, g_{02}, g_{03})}$$

[2, p. 29, 39].

§ 3. The Pauli Equation

We now derive the generalization of the Pauli equation in the general theory of relativity.

We take the velocity of a fermion to be much smaller than the velocity of light; then, without going into the mathematical details, we can write (with $a_\kappa = 0$)

$$\hat{D}_\mp^{-1} \approx \mp \frac{1}{2mc} \left\{ 1 \pm \frac{1}{2mc} \hat{D}_\pm \right\}.$$

Proceeding as in [4] (pp. 145, 146), we get

$$\left(\pm i\hbar h^\kappa_{(0)} \frac{\partial}{\partial x^\kappa} \pm \frac{1}{c} \mathfrak{Z} \pm \frac{e}{c} h^\kappa_{(0)} A_\kappa - mc \right) u_\pm$$
$$= \left\{ \frac{1}{2mc} \sum_{\alpha,\beta} C_\alpha C_\beta \sigma_\alpha \sigma_\beta \mp \frac{1}{4m^2c^2} \sum_{\alpha,\beta} C_\alpha \sigma_\alpha \left[\hat{D}_\mp, C_\beta \sigma_\beta \right]_- \right\} u_\pm. \tag{4}$$

We now define the quantum mechanical energy and momentum operators as

$$\left. \begin{aligned} \hat{E}_\pm &= \pm i\hbar c\, h^\kappa_{(0)} \frac{\partial}{\partial x^\kappa} \\ \hat{p}_\alpha &= \pm i\hbar\, h^\kappa_{(\alpha)} \frac{\partial}{\partial x^\kappa} \end{aligned} \right\}. \tag{5}$$

We can thus write (4) in the form

$$\hat{E}_{\pm} \approx mc^2 + \frac{1}{2m}\sum_{\alpha} \hat{p}_{\alpha}\hat{p}_{\alpha} \mp \mathfrak{Z}$$

$$\mp \epsilon h^{\kappa}_{(0)} A_{\kappa} \mp \frac{1}{4m^2c}\sum_{\alpha,\beta} C_{\alpha}\sigma_{\alpha}[\hat{D}_{\mp}, C_{\beta}\sigma_{\beta}]_{-} + \cdots$$

This is the required equation. It can be seen that $\mathfrak{Z}$ does in fact play the role of the spin–gravitational energy correction.

LITERATURE CITED

1. Ya. B. Zel'dovich, ZhÉTF Lett., 1, 40 (1965).
2. Ya. B. Zel'dovich and I. D. Novikov, Theory of Gravitation and the Evolution of Stars [in Russian], Nauka, Moscow (1971).
3. Contemporary Problems in Gravitation [Russian translation], Izdatel'stvo Inostrannoi Literatury (1961).
4. N. V. Mickewicz, Physical Fields in the General Theory of Relativity [in Russian], Nauka, Moscow (1969).
5. L. D. Landau and E. M. Lifshits, Classical Theory of Fields, Pergamon Press, Oxford (1962).