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DYNAMICS OF THE SYMMETRIES OF A PHYSICAL SPACE AND THEORY OF EQUIVARIANT BORDISMS

A. K. Guts and G. B. Fishkina

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The origin of the homogeneity and isotropy of the universe is analyzed on the basis of a theory of equivariant bordisms.

A real physical space or, more precisely, its geometry has several symmetries (translation, rotation, and reflection). The general theory of relativity has generated the idea of a geometry of a space which varies in time. One can thus speak in terms of the dynamics of spatial symmetries. Since a space is merely "a shadow of a four-dimensional space—time," the extension of this assertion of Minkowski to the symmetries of a space should mean that the spatial symmetries are only one of the manifestations of the symmetries of the space—time. In other words, if G is the symmetry group of the space—time then the restriction of its application in spacelike cross sections constitutes spatial symmetries. This approach to the problem of the symmetries of a space makes it necessary to appeal to the mathematical theory of G manifolds, i.e., manifolds with the action of group G [1]. It is pertinent to recall here that any Riemannian manifold with symmetry group (isometry group) G is a G manifold. At the same time, the question of the dynamics of spatial symmetries is solved on the basis of the theory of equivariant bordisms or G bordisms [2, 3]. We show below how to use these theories to solve the question of the acquisition by a space of such important properties as homogeneity and isotropy in the course of the evolution of the universe.

1. We denote by G a group and by M^n an n-dimensional, smooth, oriented manifold. We denote by $G \to \mathrm{Diff}_+(M^n)$ a homomorphism, where $\mathrm{Diff}_+(M^n)$ is the group of orientation-conserving diffeomorphisms of the manifold M^n. One then says that M^n is a G manifold and that G acts on M^n. We denote the image of an element $g \in G$ under $G \to \mathrm{Diff}_+(M^n)$ by g, and we write g(x) for the image of a point $x \in M^n$ when element g acts on it. If G is a Lie group, we also require that the mapping $G \times M^n \to M^n$, $(g, x) \to g(x)$ from our definition of the action be smooth.

We denote a G manifold M^n with a fixed action of group G by $\langle G, M^n\rangle$. Two G manifolds $\langle G, M_1^n\rangle$ and $\langle G, M_2^n\rangle$ are equivalent if there exists an orientation-conserving diffeomorphism $\varphi: M_1^n \to M_2^n$, which is an equivariant mapping, i.e., if we have $\varphi(g(x)) = g(\varphi(x))$ for all $g \in G$, $x \in M_1^n$.

Let us assume that $\langle G, M^n\rangle$ is an orientable closed, i.e., compact without an edge, G manifold. It is bordantly zero if one can find a compact, oriented G manifold $\langle G, V^{n+1}\rangle$ for which $\langle G, \partial V^{n+1}\rangle$ is equivalent to $\langle G, M^n\rangle$. Here ∂V^{n+1} is the boundary of manifold V^{n+1}. For two G manifolds $\langle G, M_1^n\rangle$ and $\langle G, M_2^n\rangle$ there exists an ordinary unconnected union $\langle G, M_1^n \cup M_2^n\rangle$. Let us assume $-\langle G, M^n\rangle = \langle G, -M^n\rangle$, where $-M^n$ is the manifold M^n with the opposite orientation. The pair $\langle G, M_1^n\rangle$ is bordant with respect to the pair $\langle G, M_2^n\rangle$ if the unconnected union $\langle G, M_1^n \cup (-M_2^n)\rangle$ is bordantly zero.

The relation of equivariant bordantness is the relation of equivalency on the set of all closed and oriented n-dimensional manifolds [2]. We denote the class of bordisms of G manifold $\langle G, M^n\rangle$ by $[G, M^n]$, and we denote the set of all such classes by $\Omega_n(G)$. The unconnected union of G manifolds induces the structure of an Abelian group in $\Omega_n(G)$. If we omit the

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requirement that the manifolds be orientable in all the definitions, we have the group of unorientable n-dimensional bordisms $N_n(G)$.

2. We treat two bordant G manifolds $\langle G, M_1^3\rangle$ and $\langle G, M_2^3\rangle$ as the initial and final states of the evolution of the symmetry group G of the physical space. The absence of symmetries G is the trivialness of the action of group G; i.e., the homomorphism $G \to \mathrm{Diff}_+(M_1^3)$ is trivial. Correspondingly, the creation of a symmetry means that the action of group G in M_2^3 is nontrivial. A breaking of symmetries G is understood in a corresponding way. In this case a finite G manifold $\langle G, M_2^3\rangle$ has a trivial action of group G. The bordism $\langle G, V^4\rangle$, whose boundary is $\partial V^4 = M_1^3 \cup (-M_2^3)$, is a space–time whose initial and final spacelike cross sections must be M_1^3 and M_2^3. Specifically, a bordism (film) $\langle G, V^4\rangle$ is equipped with a Lorentz metric of the type [4, 5].

$$g_{ik} = \gamma_{ik} - a \cdot \omega_i \otimes \omega_k, \tag{1}$$

where γ is some Riemannian metric, and ω is a 1-form for which M_1^3 and M_2^3 are spacelike three-dimensional surfaces. The Riemannian metric γ can be taken in such a way that the group G is its symmetry (isometry) group [1]. If manifold V^4 allows a space–time stratification [5] of such a nature that the layers are orbits of the group, and M_1^3 and M_2^3 are also layers, then we can assume that the given stratification has been generated by 1-form ω. In this case, however, ω is invariant under the action of group G, so the metric g allows symmetry group G. The space–time $\langle V^4, g\rangle$ becomes symmetric under group G; incidentally, so do $\langle M_1^3, g|M_1^3\rangle$, $\langle M_2^3, g|M_2^3\rangle$, where $g|M_i^3$ is a Riemannian metric which is induced on M_i^3 by Lorentz metric g. Furthermore, since M_i^3 (i = 1, 2) are orbits of group G, the corresponding physical space must be homogeneous.

The bordism $\langle G, V^4\rangle$ described above, with a Lorentz metric G which is symmetric under g, is a "Lorentz-equivariant bordism" or "Lorentz G bordism." Although the space is homogeneous (under G) in the case of a Lorentz G bordism, there are still many other problems involving the dynamics of the symmetries. For example, there is the problem of the isotropy of the space. In such a case, a finite G manifold $\langle G, M_2^3\rangle$ must have an action of group G of such a nature that a stabilizer G_x of group G at each point $x \in M_2^3$ is isomorphic with respect to the group SO(3). Understandably, the initial state $\langle G, M_1^3\rangle$ cannot have this property. This problem can be solved by a purely mathematical approach by calculating the class $[G, M_2^3]$. In the case $\langle G, M_1^3\rangle \in [G, M_2^3]$, we should say that isotropy unavoidably arises from the intial state $\langle G, M_1^3\rangle$. Furthermore, if $\Omega_3(G) = 0$ then isotropy arises from any physical state. If, on the other hand, we have $\Omega_3(G) \neq 0$, then by no means each initial state will generate isotropy, and to know the elements of group $\Omega_3(G)$ is to know the evolution paths of the symmetries of the initial states as well to determine past (present) symmetric states.

In the case $\Omega_3(G) \neq 0$ we should speak in terms of a specific type of topology and symmetry of the initial state of the universe, and we should also speak in terms of allowed types of topological structural changes which involve the space in the course of the evolution of the universe. Finally, if for G manifolds $\langle G, M_1^3\rangle$ and $\langle G, M_2^3\rangle$ there is no Lorentz G bordism, but a G bordism $\langle G, V^4\rangle$ is possible, then under the assumption that M_2^3 is an orbit of group G and that M_1^3 does not have this property we conclude that the bordism $\langle G, V^4\rangle$ represents the dynamics of a homogeneous physical space which is being created.

3. This discussion demonstrates the fundamental importance of a calculation of the groups of three-dimensional G bordisms. Unfortunately, the literature reveals little about $\Omega_3(G)$, especially in cases in which G is not a finite group. Nevertheless, we have been able to find several results, which will answer several of the questions discussed above.

There exists ([7]; [2, theorem 14.2]) a useful formula for the case of a free action of group G:

$$\Omega_n^{\text{free}}(G) \cong \Omega_{n-\kappa}(BG) \cong \sum_{p+q=n-\kappa} H_p(BG;\ \Omega_q) \bmod C, \tag{2}$$

where κ is the dimensionality of group G, BG is a classifying space for G, Ω_q is a q-dimensional group of Thom bordisms [7], C is the class of finite groups of odd order, and $H_p(A, F)$ is a p-dimensional group of homologies with coefficients in group F.

3.1 Homogeneity of Space. Here we must assume that G acts transitively on M_2^3. In such a case, however, we have $\dim G \geqslant 3$. If $G = M_2^3$, i.e., if M_2^3 is a compact Lie group which acts on itself by leftward displacements, then we have $\Omega_3(G) = 0$ [6]. Consequently, the homogeneity of a physical space (a group space; e.g., a sphere S^3 or a torus $S^1 \times S^1 \times S^1$) arises from any arbitrary, possibly completely asymmetric, initial state $\langle G, M_1^3\rangle$.

If G acts freely on 3-manifold M_2^3, we find from (2)

$$\Omega_3^{\text{free}}(G) \cong H_0(BG;\ \Omega_0).$$

Since we have $\Omega_0 \cong Z$, we have $H_0(BG; Z) \cong Z$, and thus $\Omega_3^{\text{free}}(G) = Z$. Accordingly, a free symmetry does not arise from each free symmetry if the corresponding film V^4 has no fixed points under the action of group G.

3.2. Isotropy of Space. For a semifree action of group S^3 (i.e., the stabilizer at any point either is trivial or coincides with the entire group S^3) we know that we have $\Omega_3^{\text{sfree}}(S^3) = 0$ [6]. Since we have $S^3 \cong SU(2)$, and SU(2) is locally isomorphic with respect to SO(3), the triviality of the group $\Omega_3^{\text{sfree}}(S^3)$ means that there is the possibility that the isotropy of space arises from any initial state. An enumeration of manifolds M^3 with a semifree action S^3 constitutes a study of the topological types of an isotropic physical space.

3.3. Axial Symmetry. Cylindrical symmetry can arise from an arbitrary initial state since we have $\Omega_3^{\text{sfree}}(S^1) = 0$ in the case of a semifree action of S^1 on M_2^3 [7]. In the case of a free action S^1 we know [8] that we have $[S^1, M_2^3] = 0$ in $N_3(S^1)$.

A connected compact S^1 manifold is called a "Seifert manifold." Such manifolds have been studied widely [9, 10] and have been classified. They are used to construct general 3-manifolds [11].

3.4. Discrete Symmetries. These are the symmetries which have been studied to the greatest extent in the mathematical literature. For example, we have $\Omega_3(Z_2) = 0$ [12]. Since Z_2 — the action of M^3 — is an involution P ($M^3 \to M^3$, $P^2 = id_{M^3}$), i.e., since the repeated application of P generates an identity transformation, such a symmetry as parity drops out of the Z_2 action. Consequently, if, in the course of a topological structural change, the geometry of 3-space loses parity, it must be possible for the parity to be restored after a certain time. The action of Z_2 on V^4 may be thought of as PT symmetry [provided that γ_{ik} and ω are Z_2-symmetric; see (1)]. Interestingly, we have $\Omega_4(Z_2) = Z \oplus Z \oplus Z \oplus Z_2$ [12]; i.e., PT symmetry is a "shadow" of four different types of 5-symmetries. For finite groups, Lorentz G bordisms were studied in [13]. The groups $\Omega_n(G)$, where G is a finite subgroup of SU(2), were found in [14].

3.5. Internal Symmetries. A creation of internal symmetries of physical fields in the face of a constancy of space—time apparently follows from [6], where it was shown that a space of a smooth G stratification on a closed and smooth manifold is bordantly zero.

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