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ON MAPPINGS OF FAMILIES OF SETS

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Let $Л^n$ be an n-dimensional Lobačevskiĭ space, $n \geq 2$, M be a set in $Л^n$, and let $T(Л^n)$ be a transitive subgroup of the group of motions. Let $f: Л^n \to Л^n$ be a bijective mapping possessing the following property. The image of a set $t(M)$ obtained from M with the help of an element t of the group $T(Л^n)$ is given by a set $t'(M)$, where t' is also an element of the group $T(Л^n)$.

We ask: What kind of a set should M be for the transformation f to be a motion?

To be able to formulate our results, we shall introduce certain definitions.

Let $L \equiv \{l(X): X \in Л^n\}$ denote the family of all straight lines parallel to a certain given direction, $l(X)$ being the line passing through the point X; and in this way we denote only the straight lines of the family L.

Let O be a fixed point in space. We shall call the sets obtained by rotating around the line $l(O)$, a straight line, an oricycle, and an equidistant curve, passing through the point O at an angle α, $0 \leq \alpha < \pi/2$ with respect to the line $l(O)$ and lying in a two-dimensional plane passing through the line $l(O)$, a cone $(\alpha \neq 0)$, an oricone, and an equicone respectively.

Let $L_k \equiv \{l_k(X): X \in Л^n\}$, $k = 1, \cdots, n+1$, be a family of straight lines such that a straight line $l_k(X)$ passes through the point X at an angle α_k with respect to the line $l(X)$ of the family L. We assume also that the lines $l_1(X), \cdots, l_{n+1}(X)$ are such that no n of them lie in the same hyperplane. Thus, we have $(n+1)$ different families of straight lines, and we do not exclude the case when one of them coincides with the family L.

We put

$$D(X) = \bigcup_{k=1}^{n+1} l_k(X).$$

By the pseudocenter of a hypersphere $S(X, r)$ with radius $r > 0$ and center at the point X we shall mean the point Y removed from the point X along the line $l(X)$ in the direction of parallelism of the lines of the given family by a distance $k \ln \sinh (r/k)$, where k is a constant corresponding to the Lobačevskiĭ geometry. The sphere $S(X, r)$ with pseudocenter Y we shall denote by $\Sigma(Y, r)$.

With the help of the family L one can introduce oricyclic coordinates in the space $Л^n$, and in them the metric takes the form

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$$d\sigma^2 = dy^{1^2} + \exp\left(-\frac{2y^1}{k}\right)\cdot\sum_{i=2}^{n} dy^{i^2}.$$

Let $T(Л^n)$ be a transitive subgroup of the group of motions whose elements are given in the following way:

$$t\colon (x^1,\dots,x^n) \to (\lambda(t)x^1,\ \lambda(t)x^2+\alpha^2(t),\dots,\lambda(t)x^n+\alpha^n(t)),$$
$$x^1 = \exp(y^1/k),\quad x^i = y^i/k,\quad i=2,\dots,n,$$

where $t \in T(Л^n)$ and $\lambda(t)$, $\alpha^2(t)$, $\cdots$, $\alpha^n(t)$ are numbers depending on just that element t.

Then if $C(O)$ is a set with a distinguished point O, we can consider a family of sets

$$\{C(X)\colon X \in Л^n\} \equiv T(Л^n)C(O),$$

that is $C(X)$ is the image of the set $C(O)$ under a transformation t, which is an element of the group $T(Л^n)$, if the point X is the image of the point O.

Theorem 1. *Let $Л^n$, $n \geq 2$, be an n-dimensional Lobačevskiĭ space and let $f\colon Л^n \to Л^n$ be a bijective transformation such that*

$$f[C(X)] = C[f(X)],$$

where $\{C(X)\colon X \in Л^n\}$ is a family of sets constructed in the manner described above.

The transformation f will be a motion if the set $C(O)$ is one of the following: a cone (here $n > 2$), an oricone, an equicone $(n > 2)$, a hypersphere, the set $\Sigma(O, r)$, or the set $D(O)$.

We remark, that, in the case when the equicone $C(O)$ is such that the hyperorisphere orthogonal to the line $l(O)$ at the point O has an intersection with $C(O)$ consisting of the point O only, the available proof uses the continuity property of the transformation, although it seems to us that it is superfluous. It is just that case which is important for constructing relativity theory in the De Sitter universe.

The case of the hypersphere or the set $\Sigma(O, r)$ provides an answer to the questions: What are the physical motions under which certain observations are identical in a universe with the metric

$$ds^2 = c^2\,dt^2 + \sum_{i,k=1}^{3} g_{ik}(y^1, y^2, y^3)\,dy^i\,dy^k,$$

in the cases when the front of the light propagation in 3-space is represented in the Lobačevskiĭ space by a sphere or by a set $\Sigma(X, r)$?

Theorem 2. *Let $f\colon Л^n \to Л^n$, $n \geq 2$, be a bijective transformation which carries an arbitrary oricycle into an oricycle.*

Then f is a motion.

The proofs of these theorems are quite long and cannot be presented here. The main idea consists in extending the transformation f, using the Poincaré model, onto the whole Euclidean space, and showing that it preserves the family of Euclidean cones. Then, using [1], we show that the transformation f in the Poincaré model is a composition of certain inversions and symmetries; and therefore it is a motion.

In the case of a separable Hilbert space, when instead of motions we consider affine transformations, Theorem 1 remains true with the exception of the case when one considers the set $D(O)$. One can, however, construct a generalization of the set $D(O)$, and the theorem will be valid if only the transformation is continuous. In the case of a cone the result has been known for a long time: this is the well-known theorem of A. D. Aleksandrov and V. V. Ovčinnikova [1], whose work started this line of investigations.

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