The conditions under which the number of connection components of physical space changes are determined.

In this paper, we determine the conditions under which the topology of physical space changes, more precisely, becomes disconnected. This problem was investigated in [1] for a closed universe.

Let \( M \) be a connected three-dimensional Riemannian manifold with the metric \( \gamma^{\alpha\beta}(a, \beta = 1, 2, 3) \), \( D_0 \subset M \) be a closed region, which is homeomorphic to a three-dimensional sphere. Let us suppose that in time \( t \in [0, 1] \), the number of connection components of the manifold \( M_0 = M(t = 0) \) increases, and the manifold changes into one \( M_1(t = 1) \) which is no longer connected. Figuratively speaking, a region \( D_0 \) separates from \( M_0 \). So as not to complicate the presentation, we shall assume that \( M_1 \) has two connection components \( D_1 \) and \( C_1 \), i.e., \( M_1 = D_1 \cup C_1 \), \( D_1 \cap C_1 = \emptyset \). The transition from \( M_0 \) to \( M_1 \) proceeds via some critical 3-space \( M_{1/2}(t = 1/2) \), which is obtained from \( M_0 \) by contracting the boundary \( \partial D_0 \) of the region \( D_0 \) to a point. Then, \( D_0 \) transforms into the region \( D_{1/2} \), homeomorphic to the 3-sphere \( S^3 \). Therefore, a necessary stage in the path to separation of \( D_0 \) from \( M_0 \) is stretching \( M_0 \) along \( \partial D_0 \): the transition from \( M_0 \) to \( M_{1/2} \). If \( F_0 \subset M_0 \) is an arbitrary closed two-dimensional submanifold intersecting \( D_0 \) along \( B_0 \) and, in addition, \( B_0 \) is homeomorphic to the 2-sphere, then at \( t = 1/2 \), the boundary \( \partial B_0 \) is already contracted to a point, while at \( t = 1 \), the region \( B_0 \) is separated from \( F_0 \). For this reason, we shall first study the breakdown of connectedness of the two-dimensional manifold \( F_0 \). We shall denote the manifold or space obtained from \( F_0 \) up to time \( t \) by \( F_t \).

We shall realize the separation of \( B_0 \) from \( F_0 \) as follows. We shall examine the family of Riemannian metrics \( a_{AB}(t), t \in [0, 1], A, B = 1, 2 \), defined on the manifold \( F_0 \) and satisfying the following conditions:

1) \( a_{AB}(t) \) for \( 0 \leq t < 1/2 \) belongs to class \( C^2 \) and for \( t \geq 1/2 \), the first order derivatives of the functions \( a_{AB}(t) \) are discontinuous on \( \partial B_0 \);

2) the length of the curve \( \partial B_0 \), calculated in the metric \( a_{AB}(t), t < 1/2 \), approaches 0 as \( t \to 1/2 \) or, in other words,

\[
\lim_{t \to 1/2^-} d\sigma_t|_{\partial B_0} = 0, \quad d\sigma_t|_{\partial B_0} = 0 \text{ for } t \geq 1/2,
\]

where \( d\sigma_t \) is the element of area in the metric \( a_{AB}(t) \);

3) the Riemannian spaces \( F_0 \setminus (B_0 \cup \partial B_0), B_0 \setminus \partial B_0 \) with the induced metric \( a_{AB}(t), t \geq 1/2 \), supplemented with the "point" \( \partial B_0 \), are closed oriented manifolds. We shall denote them as \( A_t \) and \( B_t \), respectively.

Let us clarify the metric conditions 1-3. We represent the transition from \( F_0 \) to \( F_1 \) through \( F_{1/2} \) on the same set of points \( F_0 \). For this, the family of topologies \( T_t, t \in [0, 1] \), is introduced on \( F_0 \) and, in addition, each topology \( T_t \) is matched with a topology generated by the metric \( a_{AB}(t) \). Therefore, the space \( F_t \) as a set equals \( F_0 \), but, in general, it has a different topology. We can write symbolically \( F_t = \langle F_0, T_t \rangle \), in particular, \( B_t = \langle B_0, T_t \rangle \), having in mind the topology induced on \( B_t \). In the topology \( T_{1/2} \), the curve \( \partial B_0 \) is a point, while the calculation of the boundary \( \partial_{1/2}B_{1/2} \) of the set \( B_{1/2} \) in the topology \( T_{1/2} \) gives \( \emptyset \), i.e., \( \partial_{1/2}B_{1/2} = \emptyset \), because \( B_{1/2} = \emptyset \) is already homeomorphic.

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to the sphere $S^2$. Thus condition 2 indicates that $\partial B_0$ is contracted into a point. The space $\mathcal{F}_{1/2}$ is a critical space; it consists of two manifolds $A_{1/2}$ and $B_{1/2}$ with the point $<\partial B_0, T_{1/2}>$. For $t > 1/2$, the manifolds $A_t$ and $B_t$ represent different connection components of the separated space $\mathcal{F}_t$ (the point $<\partial B_0, T_{1/2}>$ no longer represents two different points). There is nothing unnatural in this, since the connection components in reality are diffeomorphic (and isometric) to $A_t$ and $B_t$, respectively. Our construction is not as convenient as the Lorentz cobordism [2] between $\mathcal{F}_0$ and $\mathcal{F}_t$, but, on the other hand, it is suited for comparing the integrals taken along $\mathcal{F}_t$, $t < 1/2$ and $\mathcal{F}_s$, $s > 1/2$, which is done below.

The constructions made above permit talking about the topological metamorphosis of the manifold $\mathcal{F}_0$, due to the application of the Gauss–Bonnet theorem. This theorem says that for a two-dimensional closed oriented Riemannian manifold $\mathcal{F}$ of class $C^2$

$$\int \Gamma d\sigma = 2\pi \chi(\mathcal{F}),$$

where $\Gamma$ is the Gaussian curvature and $\chi(\mathcal{F})$ is the Euler-Poincaré characteristic.

Therefore, for $0 \leq t < 1/2$

$$\int_{\mathcal{F}_t} \Gamma d\sigma_t = 2\pi \chi(\mathcal{F}_t)$$

and for $s > 1/2$

$$\int_{\mathcal{F}_s} \Gamma d\sigma_s = 2\pi \chi(\mathcal{F}_s),$$

where $\Gamma_t$, $d\sigma_t$ are, respectively, the Gaussian curvature and the element of area in the metric $\alpha_{AB}(t)$. Let $\mathcal{F}_0$ be homeomorphic to the sphere $S^2$. Then $\chi(\mathcal{F}_0) = \chi(A_0) = \chi(B_0) = 2$. We call attention to the fact that equalities (2) were obtained as a result of condition 1, i.e., due to the loss of smoothness of the metric $\alpha_{AB}(t)$ on $\partial B_0$.

Let $V$ be a small neighborhood of the curve $\partial B_0$ (in the topology $\mathcal{T}_0$). We shall assume that $\alpha_{AB}(t) = \alpha_{AB}(0)$ outside $V$. Then, it follows from (1) and (2) that

$$\int_{V\cap A_t} \Gamma d\sigma_t - \int_{V\cap B_s} \Gamma d\sigma_s = 4\pi,$$

where $t < 1/2$, $1/2 < s$

or

$$\int_{V}(\Gamma_t d\sigma_t - \Gamma_s d\sigma_s) = 4\pi.$$  

Since $d\sigma_s = 0$ on $\partial B_0$, we obtain from (3) that there exists a neighborhood $W \subset V$ in which $\Gamma_s \geq \Gamma_t$. This means that the separation of $B_0$ from $\mathcal{F}_0$ indicates a sharp increase in curvature.

Returning now to the breakdown of connectedness of the physical space $\mathcal{M}_0$, we conclude that the separation of $D_0$ from $\mathcal{M}_0$ is characterized by a jump in the Gaussian curvature in some neighborhood $U$ of the "sphere" $\partial D_0$ for any two-dimensional closed manifold $\mathcal{F}_0$ intersecting $D_0$. From here, we conclude that there is a jump in the scalar curvature $(^s)R$ of the manifold $\mathcal{M}_0$ in some neighborhood $U \supset \partial D_0$. Indeed, $(^s)R = 2\pi + x$, where $\Gamma$ is the Gaussian curvature of the section $\mathcal{F}_0$, while $x$ is the invariant of its exterior curvature (Gauss–Codazzi equation). The section can be chosen so that $x = 0$ (for example, the section $\theta = \text{const}$ or $\varphi = \text{const}$ of the closed Friedman universe). For this reason, a jump $\delta \Gamma$ in the curvature $\Gamma$ implies a jump $\delta (^s)R$ in the curvature $(^s)R$.

Let us examine the space-time metric

$$ds^2 = (N^2 - N_t N^t) dt^2 - 2N_t dt dx^3 - \gamma_{\alpha\beta}(x, t) dx^\alpha dx^\beta,$$

on the set of events $\mathcal{M}_0 \times [0, 1]$ satisfying the conditions:

a) $t = \text{const}$ is a spacelike section with metric $\gamma_{\alpha\beta}(x, t)$;

b) $\gamma_{\alpha\beta}/\partial n$, where $n$ is the normal to the section $t = \text{const}$, are continuous;
c) \( \gamma_{AB}(x, t) = \gamma_{AB}^0 \) outside some neighborhood \( U \) of the region \( D_0 \) in the topology of the manifold \( M_0 \);

d) the metrics \( a_{AB}(t) \) induced on two-dimensional sections \( F_0 \) (they are induced by the metrics \( \gamma_{AB}(x, t) \)) satisfy the conditions 1-3 and \( \partial a_{AB}/\partial t \leq 1 \) for \( t < 1/2, \ s > 1/2 \) in \( U \);

e) the Gaussian curvature \( \Gamma \) of the section \( F_0 \) in the metric \( a_{AB}(s) \) is nonnegative (\( s > 1/2 \)).

It follows from (3) and c-d that

\[
\int_{U \cap Re_1} \Gamma_1 d^2 t \geq 4\pi + \int_{U \cap Re_2} \Gamma_0 d^2 t, \ t < 1/2, \ 1/2 < s
\]

or

\[
\langle \delta \Gamma \rangle \cdot \sigma_1 (U \cap F_0) \geq 4\pi,
\]

where

\[
\delta \Gamma = \Gamma_1 - \Gamma_0,
\]

\( \sigma_1(A) \) is the area of the region \( A \subset F_0 \) in the metric \( a_{AB}(t) \), and

\[
\langle f \rangle = \frac{1}{\sigma_1(A)} \int f d\sigma_1
\]

is the integral average of the quantity \( f \).

The dynamics of the 3-geometry is described by the Einstein equations, from which follows ([3], p. 157)

\[
(\delta R) \cdot K_{2,t} = \frac{16\pi G}{c^4} \varepsilon(t), \ t \in [0, 1];
\]

\[
K_{2,t} = (K_{AB}^0(t))^2 - K_{AB}^0(t) K_{AB}^0(t),
\]

where \( K_{AB}(t) \) is the tensor of the exterior curvature of the section \( t = \text{const}. \)

Then

\[
\langle \delta (\delta R) \rangle + \langle \delta K_2 \rangle = \frac{16\pi G}{c^4} \langle \delta \varepsilon \rangle,
\]

where

\[
\delta (\delta R) = (\delta R) - (\delta R), \ \delta K_2 = K_{2,t} - K_{2,t}, \ \delta \varepsilon = \varepsilon(s) - \varepsilon(t), \ t < 1/2, \ 1/2 < s.
\]

But, as demonstrated above,

\[
\langle \delta (\delta R) \rangle \sim 2 \langle \delta \Gamma \rangle,
\]

where \( \Gamma \) is the Gaussian curvature of the two-dimensional section \( F_0 \). At the same time, due to the condition b above, the exterior curvature \( K_{2,t} \) will be a continuous function on \( M_0 \times [0, 1] \). Therefore

\[
\langle \delta K_2 \rangle = (K_{2,t} - K_{2,t}) \big|_{t-x_0, (t, s)} \rightarrow 0.
\]

For this reason, for some \( t_0 < 1/2, \ 1/2 < s_0 \), the quantity \( \langle \delta K_2 \rangle \) is negligibly small and then, from (4)-(8), we obtain

\[
\langle \delta \varepsilon \rangle \geq \frac{c^4}{2\pi G \sigma_1 (U \cap F_0)}
\]

It is now entirely permissible to write

\[
\langle \delta \varepsilon \rangle \geq \frac{c^4}{2\pi G \sigma},
\]

where \( \sigma \) is the characteristic section of the region \( D_0 \).
Equation (9) gives us the average value of the jump in the energy density, which gives rise to separation of the region $D_0$.

From (9), we obtain the following estimates:

1) $\sigma \sim 10^{20} \text{ cm}^2$ (sun), $<\delta \rho > = <\delta \epsilon >/c^2 \sim 10^7 \text{ g/cm}^3$;

2) $\sigma \sim 10^{12} \text{ cm}^2$ (neutron star), $<\delta \rho > \sim 10^{15} \text{ g/cm}^3$;

3) $\sigma \sim 10^{-66} \text{ cm}^2$ (singularity), $<\delta \rho > \sim 10^{93} \text{ g/cm}^3$.

Thus separation of small regions is inhibited by a strong potential barrier. Motion induced in space by a change in the topology of the space itself will require enormous expenditures of energy. The parameters of superdense configurations are close to those for separation from space. This confirms our conclusions, obtained in [1] for a closed model of the universe. Breakdown of connectedness is to be expected in gravitational collapse of massive stars because in this case singularities arise (based on Penrose's theorems [4], p. 242), which entail a singularity of the curvature. It is easy to see that the above picture of the breakdown of connectedness is in many ways similar to the process of gravitational self-closure accompanied by gravitational collapse of homogeneous spherically symmetrical configurations, analyzed in detail in [5] (p. 52). For this reason, it may be expected that singularities form due to breakdown of the connectedness of 3-space.

LITERATURE CITED


THEORY OF SPATIALLY PERIODIC STRUCTURES.

BOSE EXCITATION GREEN'S FUNCTIONS

A. I. Olemskoi

We discuss Green's function techniques in the description of spatial ordering viewed as a Bose-Einstein condensation of the density wave of the ordering units.

1. Two approaches can be used in treating spatial ordering in quantum statistics [1, 2]. The first is based on the exclusion principle, according to which units forming the spatially periodic structure (the atoms of a crystallizing liquid or solid solution or the phase separations in a quasiperiodic macrostructure of dissociating alloys) cannot occupy the same spatial position $\mathbf{r}$. This allows the representation of the ordering process as a redistribution of fermions over the states $\mathbf{r}$. The corresponding Green's function formalism is identical in form to the techniques of Gor'kov in the theory of superconductivity, and has been discussed in [1].

In the second approach, the ordering process is thought of as a redistribution of the Bose density of the ordered structure over values of the wavevector $\mathbf{k}$. The condition that this method be applicable is that the Bose amplitudes $C_k$ be statistically independent for different values of $\mathbf{k}$ [3]. However it can easily be shown that if the total number of structural units is conserved, the $C_k$ satisfy the relation

$$\sum_{\mathbf{k}} <|C_\mathbf{k}|^2> = \text{const}, \quad (1)$$

*The proof of (1) is carried out in similar fashion to the case of an ordered solid solution [4], where const $= C(1 - C)N$, $C$ is the concentration, and $N$ is the total number of atoms.