A CHARACTERIZATION OF TWO-DIMENSIONAL ELEMENTARY GEOMETRIES

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Consider a metric space M with metric ρ and assume that it has the following property: for each point $x \in M$ there exists a spherical neighborhood (ball) of x, $B(x, \delta_x)$, $\delta_x > 0$, which admits a rotation in the sense of Buseman, i.e., for any points a, a', b, $b' \in B(x, \delta_x)$ such that $\rho(xa) = \rho(xa')$, $\rho(xb) = \rho(xb')$, and $\rho(ab) = \rho \times (a'b')$, there is an isometry map of the ball $B(x, \delta_x)$ onto itself which keeps x fixed and takes a into a' and b into B'. Our problem is: when is the universal covering space of M isometric to the Euclidean plane, to the Lobachevski plane, or to the two-dimensional sphere?

This problem may be interpreted as a local version of the well-known Helmholtz–Lie conjecture. The most satisfactory solution of the latter is given in Freudenthal's paper [1], where the following result was obtained.

Let M be a connected, locally compact metric space, and let F be a doubly transitive group of homeomorphisms of M (i.e., given two arbitrary pairs of points in M, there is an element of F which takes the first pair into the second). Moreover, let M and F satisfy the following axioms:

(S) given two arbitrary closed subsets of M, A and B, $A \cap B = \varnothing$, there is an open subset U of M, $U \neq \emptyset$, such that for any $\lambda \in F$ either $\lambda(\bar{U}) \cap A = \varnothing$, or $\lambda(U) \cap B = \varnothing$;

(V) the group F is complete;

(Z) let J_{x_0} be the isotropy subgroup of F at a point $x_0 \in M$. Then there is an orbit $J_{x_0}(y)$ which separates the space M.

Then M is already a doubly transitive homogeneous space in the sense of Birkhoff–Wang, and, in particular, M is a Euclidean space, a hyperbolic space, or a sphere, while F is a closed subgroup of the corresponding group of isometries.

Analogous results were obtained by others [2-6]. The typical method used in these papers is global and relies mainly on results from theory of Lie groups. However, the following local variant of the solution to the Helmholtz–Lie problem, due to Buseman, is known.

THEOREM B. Let $\langle M, \rho \rangle$ be a G-space such that each point $x \in M$ has a spherical neighborhood $B(x, \delta_x)$, $\delta_x > 0$, which admits a rotation in the sense of Buseman. Then the universal covering space of M is elementary, i.e., Euclidean, hyperbolic or a sphere ([2], p. 411).

Recall that according to Buseman a G-space is any space which satisfies the following axioms:

I. $\langle M, \rho \rangle$ is a metric space.

II. $\langle M, \rho \rangle$ is boundedly compact, i.e., any bounded infinite subset has at least a limit point.

III. Given arbitrary distinct points x, z, there is a point y such that (xyz), i.e., $y \neq x$, $y \neq z$ and $\rho(xy) + \rho(yz) = \rho(xz)$.

IV. Given any point $x \in M$, there is a positive number ρ_x such that for arbitrary distinct points, y, z in the open ball $B(x, \rho_x)$ there is a point u such that (yzu).

V. If (xyz_1), (xyz_2), and $\rho(yz_1) = \rho(yz_2)$, then $z_1 = z_2$.

Although Buseman's result is the only attempt known by the author to solve the Helmholtz–Lie problem from the local point of view, it does not give a complete answer to the problem stated at the beginning of this paper. Indeed, the local uniqueness if the geodesic joining two points x, $y \in B(p, \rho_p)$ is a direct consequence of axioms IV and V, and this is such a strong assumption that it should be dropped. We remark that in introducing

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the notion of a G-space, more general questions than that of finding a direct solution to the Helmholtz–Lie problem were posed. Below we discuss a system of axioms whose specific purpose is to lead to an answer of the problem formulated at the beginning of our paper.

From now on, M shall denote a separable, locally compact, metric space with intrinsic metric ρ.

Denote by r(x) the lowest upper bound of all numbers $r > 0$ such that the open ball $B'(x, r)$ is a compact subset of M. Further, let p(x) be the lowest upper bound of all numbers $p_x > 0$ such that given $y, z \in B(x, p_x)$, then y is joined to z by a geodesic. It is known that either $p(x) = +\infty$ for all $x \in M$, or the function p(x) is continuous and takes only finite values.

We now formulate two axioms.

(A_1) Given any point $x \in M$, there is $d(x) > 0$ such that if I(x) is the group of isometries of the ball B(x, d(x)) onto itself, then I(x) acts effectively* and transitively on each sphere S(x, r), $0 < r < d(x)$, and $\lambda(x) = x$ for all $\lambda \in I(x)$.

(A_2) Given any point $x \in M$, there is $\delta_x > 0$ such that $\delta_x < \min(d(x), r(x), p(x))$ and the following holds:

a) the sphere S(x, r) is connected for all r, $0 < r \le \delta_x$;

b) on each sphere S(x, r), $0 < r \le \delta_x$ there exist two distinct points a_r, b_r which separate S(x, r), i.e., $S(x, r)\setminus\{a_r, b_r\} = A_1 \cup A_2$, where $A_1 \cap A_2 = \varnothing$, and A_1, A_2 are open nonempty subsets of $S(x, r)\setminus\{a_r, b_r\}$;

c) given any r, $0 < r \le \delta_x$, there is an isometry $\lambda \in I(x)$ such that $\lambda(a_r) \in A_1$ and $\lambda(b_r) \in A_2$ [or $\lambda(a_r) \in A_2$, $\lambda(b_r) \in A_1$].

We remark that axiom (A_2) is a local variant of the axioms (S) and (Z) introduced by H. Freudenthal. Here the role of condition b) is that of fixing the dimension of spheres: we deal with one-dimensional spheres, and hence with a two-dimensional space M.

THEOREM 1. Let the space M satisfy axioms (A_1) and (A_2). Then

(1) S(x, r) is homeomorphic to the one-dimensional Euclidean sphere, for all $x \in M$ and $0 < r \le \delta_x$.

(2) The group I(x) is a compact one-dimensional Lie group, and all its isotropy subgroups are compact and zero-dimensional.

(3) The connected component of the identity of I(x) acts effectively and transitively on S(x, r), $0 < r \le \delta_x$, and is isomorphic to the Lie group SO(2).

THEOREM 2. Any space M satisfying axioms (A_1) and (A_2) is a two-dimensional topological manifold.

We need three more axioms in order to solve the problem.

(A_3) Given any point $x \in M$, the ball B(x, d(x)) admits a Buseman rotation.

(A_4) Given any point $x \in M$, there is a point $y \in M$ such that $\rho(xy) < \min(d(x), d(y), \delta_y)$.

(A_5) Given any point $x \in M$, and any two geodesic starting from x, the angle between these geodesics exists in the sense of A. D. Aleksandrov. Moreover, there are at least two geodesics starting at x between whom the angle is not zero.

Axioms (A_3) and (A_4) are just a strengthening of axiom (A_1). At the same time, axiom (A_4) eliminates from consideration spaces with multi-faced metrics, which are not elementary.

Definition. A space M which satisfies axioms (A_1)–(A_5) is called an r-space.

THEOREM 3. Let M be an r-space. Then for each $x \in M$ one can find a number η_x, $0 < \eta_x \le \delta_x$, such that any point $y \in B(x, \eta_x)$ may be joined to x by a unique geodesic.

This theorem is the key result in all our study. Its proof is quite complicated and lengthy (Sec. 4).

The next theorem gives the answer to the problem solved in this note.

THEOREM 4. Let M be a complete r-space. Then M is a two-dimensional Buseman G-space, whose universal covering space is elementary, i.e., the Euclidean plane, the Lobacevski plane, or the sphere.

*If the action of I(x) on S(x, r_0), $0 < r_0 < d(x)$, $d(x) \le p(x)$, is not effective, then the geodesic joining some point $y \in B(x, d(x))$ with x is not unique (V. N. Berestovskii).

COROLLARY. Any complete r-space is one of the following spaces: a sphere, a projective space, a Euclidean plane, a cylinder, a torus, a Möbius band, a Klein bottle, or, finally, a two-dimensional locally hyperbolic space; there is an infinite number of spaces of the last type, but their description is known [7].

Remark. An r-space which is not complete is not a Buseman G-space. Moreover, there are incomplete r-spaces whose universal covering spaces are not isometric to elementary ones: consider the punctured Euclidean space.

The results in this paper were previously announced in [8].

Finally, I would like to thank V. A. Zalgaller and V. N. Berestovskii for their valuable remarks and help.

1. NOTATIONS

In this paper we use the following notations:

$\rho(a, b)$) the distance between points a, b; ab) a geodesic with extremities a and b; S(L)) the length of the curve L; $L|[a, b]$) the piece of curve L between the points $a, b \in L$;

$B(a, r)$) the open ball with center a and radius $r > 0$; $B'(a, r)$) the closed ball; $S(a, r)$) the sphere with center a and radius $r > 0$;

$\overline{\overline{A}}$) the cardinal of the set A; $\mathrm{ord}_\alpha A$) the order of the topological space A at the point α [9];

S^1) the unit circle in the Euclidean plane; $X \approx Y$) spaces X, Y are homeomorphic; $G \cong H$) groups G and H are isomorphic; int A) the interior of the set A; $\bar{A}$) the closure of the set A; Fr (A)) the boundary of the set A.

2. PROOF OF THEOREM 1

(1) Choose a number r, $0 < r \le \delta_X$, and let a, b separate $S(x, r)$, as guaranteed by condition b) of axiom (A_2):

Write $\Sigma = S(x, r)\backslash\{a, b\} = A_1 \cup A_2$, $A_1 \cap A_2 = \varnothing$, where A_1, A_2 are nonempty open subsets of Σ. Since Σ is open in $S(x, r)$, A_1, A_2 are open in $S(x, r)$. Furthermore, since $S(x, r)\backslash A_1 = A_2 \cup \{a, b\}$, then $\bar{A}_2 \subset A_2 \cup \{a, b\}$. Consequently, $\mathrm{Fr}(A_2) \subset \{a, b\}$. We remark that here the closure and boundary are taken relative to $S(x, r)$. Now let $p \in A_1$, $q \in A_2$ be arbitrary points. Then $\bar{A}_1 \cap \{q\} = \varnothing$ and $\overline{\overline{\mathrm{Fr}(A_1)}} \le 2$, i.e., $\mathrm{ord}_{p,q} S(x, r) \le 2$. According to axiom (A_2), condition c), there is a $\lambda \in I(x)$ such that $\lambda(a) \in A_1$, $\lambda(b) \in A_2$. Therefore, we get $\mathrm{ord}_{\lambda(a),\lambda(b)} S(x, r) \le 2$. Since λ is an isometry, $\mathrm{ord}_{a,b} S(x, r) \le 2$.

Now let $p \in S(x, r)$, $p \ne a$, b, and assume that $p \in A_1$. Then there is $\varepsilon > 0$ small enough to ensure that $B(p, \varepsilon) \cap S(x, r) \subset A_1$, and that $B(a, \varepsilon) \cap A_2 \ne \varnothing$, $B(b, \varepsilon) \cap A_2 \ne \varnothing$. We claim that the point p cannot be isolated: assuming the contrary, axiom (A_1) would imply that the sphere consists only of isolated points, and, being compact, the number of these points is finite and each one is an open subset of the sphere. But this contradicts the connectedness of the sphere. Therefore, there are points a_ε, b_ε such that $a_\varepsilon \in B(a, \varepsilon) \cap A_2$, $b_\varepsilon \in B(b, \varepsilon) \cap A_2$, and there are isometries $\lambda, \lambda' \in I(x)$ such that $\lambda(a) = a_\varepsilon$, $\lambda'(b) = b_\varepsilon$. Then $\lambda(p), \lambda'(p) \in B(p, \varepsilon) \cap A_1$. As we proved above

$$\mathrm{ord}_{\lambda(p),\lambda(a)} S(x, r) \leqslant 2, \quad \mathrm{ord}_{\lambda'(p),\lambda'(b)} S(x, r) \leqslant 2.$$

Since λ, λ' are isometries, $\mathrm{ord}_{p,a} S(x, r)$, $\mathrm{ord}_{p,b} S(x, r) \leqslant 2$.

In conclusion, the inequality

$$\mathrm{ord}_{p,q} S(x, r) \leqslant 2$$

holds for all points $p, q \in S(x, r)$.

By Theorem 9 of ([9], p. 291),

$$\mathrm{ord}_p S(x, r) \leqslant 2,$$

for all points $p \in S(x, r)$, i.e., $S(x, r)$ is a regular space. Let us show that $\mathrm{ord}_p S(x, r) = 2$. Assume the contrary, i.e., that there is a point p_0 such that $\mathrm{ord}_{p_0} S(x, r) \le 1$. Then axiom (A_1) shows that $\mathrm{ord}_q S(x, r) \le 1$ for all points $q \in S(x, r)$. Now, by Theorem 1 of ([9], p. 295), the set $S(x, r)\backslash\{a\}$ is connected. But $b \in S(x, r)\backslash\{a\}$ and $\mathrm{ord}_b(S(x, r)\backslash\{a\}) \leqslant 1$ (Theorem 3 of [9], p. 283). Consequently, $S(x, r) : \{a, b\}$ is connected, which contradicts condition b) of axiom (A_2).

Therefore, $\text{ord}_p S(x, r) = 2$. Now, Theorem 6 of ([9], p. 299) or Theorem 8" of ([9], p. 302) shows that $S(x, r) \approx S^1$.

Step 1 is proven.

(2) The ball $B(x, d(x))$, being a subset of a separable, locally compact space with an intrinsic metric, is a connected, separable, locally compact metric. Since $I(x)$ is the isotropy subgroup of the group of all isometrics of $B(x, d(x))$ onto itself, it can be endowed with the compact open topology. Thus, $I(x)$ becomes a compact continuous group of transformations of the ball $B(x, d(x))$ ([10], Theorem 2.5, Chap. IV).* The group $I(x)$ acts effectively and transitively on the sphere $S(x, \delta_X)$, which is a one-dimensional manifold, homeomorphic to S^1. Consequently, $I(x)$ is a Lie group ([11], Theorem 75) of dimension 1 [12]. We see that any isotropy subgroup of $I(x)$ is compact and zero-dimensional.

(3) Let K be an isotropy subgroup of $I(x)$. Then the quotient space $X = I(x)/K$ has a unique analytic structure such that $I(x)$ is a Lie group of transformations of the manifold X ([10], p. 132). But $X \approx S(x, \delta_X)$, and so X is compact, connected and one-dimensional. As such X is diffeomorphic to S^1. We obtain that the connected component $I_0(x)$ of the identity of $I(x)$ acts effectively and transitively on $X \approx S(x, r)$, $0 < r \le \delta_X$ ([10], p. 132]. Since the Lie group $I_0(x)$ is connected, one-dimensional, and compact, $I_0(x) = SO(2)$.

Theorem 1 is proven.

3. PROOF OF THEOREM 2

1. Proposition 1. There is no isometry $\lambda \in I_0(x)$, different from the identity which keeps fixed some point $y \in S(x, r)$, $0 < r \leqslant \delta_x$.

Indeed, let $\lambda \in I_0(x)$, such that λ is not the identity, but satisfies $\lambda(y) = y$ for some $y \in S(x, r)$. Fix an orientation of the sphere $S(x, r)$, i.e., a direction on the closed Jordan curve $S(x, r)$. Let $\varepsilon > 0$ be small enough, such that $\varepsilon < \rho(a_r b_r)$, where a_r, b_r are the points discussed in condition b) of axiom (A_2) and the sphere $S(y, \varepsilon)$ is compact, $S(y, \varepsilon) \approx S^1$. Then $S(y, \varepsilon) \cap S(x, r)$ is a nonempty compact subset of M. Now choose points z_1, z_2 such that if we write $L_1(y, z_1)$, $L_2(y, z_2)$ for the compact arcs of the sphere $S(x, r)$ with extremities y, z_1 and y, z_2, respectively, and which are contained entirely in the closed ball $B'(y, \varepsilon)$, then $L_i(y, z_i) \cap [S(y, \varepsilon) \cap S(x, r)] = \{z_i\}$ $(i = 1, 2)$, and $(L_i(y, z_i)\setminus\{z_i\}) \cap [S(y, \varepsilon) \cap S(x, r)] = \varnothing$. Then two cases are possible: $\lambda(z_1) = z_1$ or $\lambda(z_1) = z_2$. The second possibility must be eliminated, because λ does not preserve the orientation of $S(x, r)$, and this contradicts the connectedness of the group $I_0(x)$. Considering the first case, we get that, in general, the point z_1 arbitrarily close to y is also fixed. Since the sphere $S(x, r)$ is compact, the latter implies $\lambda(z) = z$ for all $z \in S(x, r)$. This contradicts our hypothesis that λ is not the identity of $I_0(x)$, while $I_0(x)$ acts effectively on each sphere $S(x, r)$, $0 < r \le \delta_X$ [which shows that only the identity of $I_0(x)$ can be the identity isometry on $S(x, r)$].

This completes the proof of Proposition 1.

2. Proof of Theorem 2. Let $y_0 \in S(x, \delta_x)$ and let L be any geodesic joining x to y_0. We parametrize the group $I_0(x)$ by a parameter φ, $0 \le \varphi < 2\pi$, i.e., we choose a continuous bijective map $\mathscr{L} : [0, 2\pi) \to I_0(x)$, which is possible because $I_0(x) \cong SO(2)$. Obviously, given any $\varphi \in [0, 2\pi)$, $\mathscr{L}(\varphi)(L)$ intersects the sphere $S(x, r)$, $0 < r \le \delta_X$, at a unique point. Now pick $z \in B'(x, \delta_x)$. Then one can find $\varphi_z \in [0, 2\pi)$ such that $\{z\} = \mathscr{L}(\varphi_z)(L) \cap S(x, \rho(xz))$. The number φ_Z is uniquely determined by z: if $\varphi' \ne \varphi_Z$ and φ' has the same property as φ_Z, i.e., $\{z\} = \mathscr{L}(\varphi')(L) \cap S(x, \rho(xz))$, then $z = \mathscr{L}(\varphi')(z_0) = \mathscr{L}(\varphi_z)(z_0)$, where $\{z_0\} = L \cap S(x, \rho(xz))$. But then $[\mathscr{L}(\varphi')]^{-1} \circ \mathscr{L}(\varphi_z)(z_0) = z_0$, while the isometry $[\mathscr{L}(\varphi')]^{-1} \circ \mathscr{L}(\varphi_z)$ is not the identity of $I_0(x)$. This contradicts Proposition 1.

Let K_E be the closed disk in the Euclidean plane E^2 defined in polar coordinates (φ, r) by the relations $0 \leqslant \varphi < 2\pi$, $0 \leqslant r \leqslant \delta_x$. Define a map $f: B'(x, \delta_X) \to K_E$ by $f(z) = (\varphi_Z, \rho(xz))$, $z \in B'(x, \delta_X)$.

Since $I_0(x)$ acts transitively on every sphere $S(x, r)$, $0 < r \le \delta_X$, f is defined on the entire ball $B'(x, \delta_X)$. According to Proposition 1, f is one-to-one. Now let us show that f is surjective, i.e., that it maps $B'(x, \delta_X)$ onto K_E. If $(\varphi, r) \in K_E$, then, obviously, $z = \mathscr{L}(\varphi)(L) \cap S(x, r) \in B'(x, \delta_x)$ and $f(z) = (\varphi, r)$. Consequently, f is a bijection from $B'(x, \delta_X)$ onto K_E. It remains to check that f is a homeomorphism. Now f^{-1} maps the compact set K_E onto the Hausdorff space $B'(x, \delta_X)$, and so it suffices to show that f^{-1} is continuous. Let $(\varphi_n, r_n) \to (\varphi, r)$ as $n \to \infty$. Set $z_n = \mathscr{L}(\varphi_n)(L) \cap S(x, r_n)$, $z = \mathscr{L}(\varphi)(L) \cap S(x, r)$. Then $z_n, z \in B'(x, \delta_x)$ and $f^{-1}(\varphi_n, r_n) = z_n$, $f^{-1}(\varphi, r) = z$. Since $\mathscr{L}(\varphi_n) \to \mathscr{L}(\varphi)$ as $n \to \infty$, $\mathscr{L}(\varphi_n)(L) \to \mathscr{L}(\varphi)(L)$ as $n \to \infty$ (here the convergence is understood as follows: the geodesics $\mathscr{L}(\varphi_n)(L)$ and $\mathscr{L}(\varphi)(L)$ are parametrized by a parameter $t \in [0, 1]$, and $\mathscr{L}(\varphi_n)(L)(t) \to \mathscr{L}(\varphi)(L)(t)$ as

*In Theorem 2.5 of [10] the space considered is a Riemann space. However, the proof carries over, without changes, to a connected separable metric space.

$n \to \infty$). Analogously $S(x, r_n) \to S(x, r)$ as $n \to \infty$ [the parametrizations of $S(x, r_n)$ and $S(x, r)$ are given by $\mathscr{L}(\varphi)$]. Consequently, $z_n \to z$ as $n \to \infty$. In other words, f^{-1} is continuous, f is a homeomorphism, and so M is a two-dimensional topological manifold. Theorem 2 is proven.

4. PROOF OF THEOREM 3

1. LEMMA 1. If $B(p, r_p)$ and $B(q, r_q)$ admit (Buseman) rotations and $\rho(pq_1) = \rho(pq) < r_p$, then $B(q_1, \min \times (r_q, r_p - \rho(pq))$ admits a rotation.

A proof is given in ([2], p. 408, Theorem (48.1)).

LEMMA 2. Let $x \in M$. There is a number ε, $0 < \varepsilon < \delta_x$, such that the ball $B(u, \varepsilon)$ admits a rotation for all points $u \in B(x, \varepsilon)$.

Proof. In Lemma 1, take q = x and p = y, where y is the point corresponding to x by virtue of axiom (A_4). By Lemma 1, the ball $B(v, \min(d(x), d(y) - \rho(xy))$ admits a rotation for any point $v \in S(y, \rho(xy))$. Now, let $a, b \in S(y, \rho(xy))$ be points which separate the sphere $S(y, \rho(xy))$ [condition b), axiom (A_2)]. Since $\rho(xy) < \delta_y$, $S(y, \rho(xy)) \approx S^1$, and hence $S(x, r) \cap S(y, \rho(xy))$ is a nonempty set for all r, $0 < r \leqslant \rho(ab)$. Let $\tilde{\varepsilon} = \min(d(x), \delta_y - \rho(xy), \rho(ab))$ and $\beta_u = \min(d(x), \tilde{\varepsilon} - \rho(xu), \rho(ab))$, where $u \in B(x, \tilde{\varepsilon})$ is arbitrary. Since $\tilde{\varepsilon} \leqslant \rho(ab)$, one can find a point $v \in S(x, \rho(xu)) \cap S(y, \rho(xy))$. Denote by $\lambda \in I(x)$ a rotation such that $\lambda(x) = x$, $\lambda(u) = v$. Then

$$\lambda[B(u, \beta_u)] \subset B(v, \min(d(x), d(y) - \rho(xy)).$$

This shows that the ball $B(u, \beta_u)$ admits a rotation. Indeed, let $a, a', b, b' \in B(u, \beta_u)$ be the points appearing in the definition of the rotation. Then a rotation μ of the ball $B(u, \beta_u)$, which takes a into a' and b into b' is given by $\mu = \lambda^{-1} \circ \theta \circ \lambda$, where θ is a rotation of the ball $B(v, \min(d(x), d(y) - \rho(xy)))$ taking $\lambda(a)$ into $\lambda(a')$ and $\lambda(b)$ into $\lambda(b')$. Now take $\varepsilon = \tilde{\varepsilon}/2$. If $\rho(xu) < \varepsilon$, then $\beta_u \geq \varepsilon$, i.e., the ball $B(u, \varepsilon)$ admits a rotation. Lemma 2 is proven.

2. Proof of Theorem 3. Assume that the theorem is not true. Then there is a point $x \in M$ with the property that given any η, $0 < \eta \leq \delta_x$, one can find a point $y \in B(x, \eta)$ which can be joined to x by more than one geodesic. Denote by $(xy)_i$, $i \in I$, the geodesics joining x to y, where I is a set indexing all such geodesics.

(a) Let us show that I is an infinite set. If not, i.e., if $1 < \bar{I} < \aleph$ and $\bar{\bar{I}} = m$, m a natural number, we let $i_1 \neq i_2$, $i_1, i_2 \in I$, and let η' be a number such that $\eta' \leqslant \delta_x$, $\eta' < \rho(xy)$, and $(xy)_{i_1} \cap S(x, \eta') = \{z\}$, $(xy)_{i_2} \cap S(x, \eta') = \{w\}$, $z \neq w$. Then there are at most $m - 1$ geodesics from the family $\{(xy)_i\}_{i \in I}$ which pass through z. Moreover,

$$\overline{\overline{\{(xz)_j\}}}_{j \in J} = \bar{J} \leqslant m - 1.$$

which, as one can easily see, contradicts the assumption made at the onset of this proof. Indeed, let L' be an m-th geodesic, different from the geodesics of the family $\{(xz)_j\}_{j \in J}$ and joining x and z, and let $L'' = (xy)_{i_1} | [z, y]$ be a geodesic with extremities z and y. Then $L = L' \cup L''$ is a geodesic with ends x and y. But $L \notin \{(xy)_i\}_{i \in I}$, since the piece L' of this geodesic is "new" [i.e., it is not the restriction of a geodesic from the family $\{\widetilde{(xy)_i}\}_{i \in I}$]. Contradiction.

(b) By Lemma 2, there is $\varepsilon > 0$ ($\varepsilon < \delta_x$) such that the ball $B(u, \varepsilon)$ admits a rotation, for all $u \in B(x, \varepsilon)$. Choose a radius μ_x such that the ball $B(x, \mu_x)$ is homeomorphic to an open disk in the Euclidean plane, and set $\varepsilon_0 = \min(\mu_x, \varepsilon/3)$, $U = B(x, \varepsilon_0)$. Then any ball $B(u, \varepsilon_0)$, $u \in U$, admits rotations, and all such rotations are defined on the entire space U.

There are two geodesics $(xy)_{i_1}$ and $(xy)_{i_2}$ such that one can find two distinct points $p, q \in (xy)_{i_1} \cap (xy)_{i_2}$ enjoying the property

$$[(xy)_{i_1} | [p, q]] \cap [(xy)_{i_2} | [p, q]] = \{p, q\},$$

i.e., these geodesics do not have common points between p and q.

With no loss of generality, one may assume that $p, q \in U$. Set

$$L_1 = (xy)_{i_1} | [p, q], \; L_2 = (xy)_{i_2} | [p, q], \; s = s(L_1) = s(L_2) > 0.$$

Let σ be the closed subdomain of U, bounded by geodesics L_1 and L_2. Obviously, σ is homeomorphic to a closed disk in the Euclidean plane.

Consider the balls $B(p, s/2)$, $B(q, s/2)$. There is no point $w \in \operatorname{int} \sigma$ such that $w \in B(p, s/2)$ and $w \in B'(q, s/2)$, or $w \in B'(p, s/2)$ and $w \in B(q, s/2)$: if this were not true, then there would exist a rectifiable curve L with extremities p and q, whose length was strictly smaller than s, which is impossible.

Therefore, we have at most

$$\sigma \cap B'(p, s/2) \cap B'(q, s/2) = S(p, s/2) \cap S(q, s/2) \cap \sigma. \tag{1}$$

Moreover, the following equality holds

$$[B'(p, s/2) \cup B'(q, s/2)] \cap \sigma = \sigma. \tag{2}$$

Indeed, it suffices to show that through each point $w \in \sigma$ passes a geodesic which joins p to q and is entirely contained in σ. If $w \in \mathrm{Fr}(\sigma)$, then $w \in L_1$ or $w \in L_2$. Now let $w \in \mathrm{int}\,\sigma$, and choose a point $v \in L_1$ such that $\rho(pv) = \rho(pw)$. Such a point exists, because $w \in B'(p, s)$. Then there is a rotation $\lambda \in I(p)$ with $\lambda(v) = w$. The geodesic $\lambda(L_1)$ intersects L_1 only at the point p (see Proposition 1), but it certainly intersects L_2. We replace the pieces of the geodesic $\lambda(L_1)$ which lie outside σ by the corresponding pieces of the geodesic L_2. If u is the last point of intersection of $\lambda(L_1)$ and L_2 [when we move from p towards $\lambda(q)$], then the piece $\lambda(L_1)|[u, \lambda(q)]$ is replaced by $L_2|[u, q]$. As a result of these modifications, we get a geodesic $pq \subset \sigma$ passing through w. Therefore, (2) is true.

(c) The spheres S(p, s/2) and S(q, s/2) have a common arc τ, which is a geodesic.

Indeed, $S(p, s/2) \approx S^1$, $S(q, s/2) \approx S^1$, and hence equalities (1), (2) show that

$$\tau = S(p, s/2) \cap S(q, s/2) \cap \sigma$$

is an arc homeomorphic to the segment [0, 1] of the real line. Let $a, b \in \tau$, $a \neq b$, be points close enough, such that all geodesics $\{(ab)_i\}_{i \in I}$ are contained in σ. One can assume, with no loss of generality, that there are geodesics $(ab)_{i_1}$, $(ab)_{i_2}$ satisfying

$$(ab)_{i_1} \cap (ab)_{i_2} = \{a, b\} \tag{3}$$

and

$$(ab)_{i_1} \subset B'(p, s/2), \; (ab)_{i_2} \subset B'(q, s/2). \tag{4}$$

Indeed, if (3) does not hold, then one can choose $a' \in (ab)_{i_1}$, $b' \in (ab)_{i_2}$ which satisfy (3), and replace a, b by a', b'. To verify (4), let, for example, $(ab)_{i_1}, (ab)_{i_2} \subset B'(p, s/2)$. There is a rotation $\lambda \in I(a)$, with $\lambda(p) = q$ and $\lambda(q) = p$. Now let $\mu \in I(q)$ be a rotation satisfying $\mu(\lambda(a)) = a$, $\mu(\lambda(b)) = b$. Then the isometry $\mu \circ \lambda$ maps B'(p, s/2) onto B'(q, s/2), and takes a into a, b into b. Consequently, $(\mu \circ \lambda)\left[(ab)_{i_1}\right] \subset B'(q, s/2)$ is a geodesic joining a and b, and so (4) is true.

Set $K_1 = (ab)_{i_1}$, $K_2 = (ab)_{i_2}$, and let σ_1 be the closed subdomain bounded by K_1 and K_2. Clearly, $\sigma_1 \subset \sigma$ and $\tau_1 = \tau|[a, b] \subset \sigma_1$. Let $z_1 \in \tau_1$. Then there is a rotation $\lambda \in I(a)$ such that $z_1 \in \lambda(K_1)$. Let z' be the next point of intersection of the curves $\lambda(K_1)$ and K_2 after z_1 [when we move on $\lambda(K_1)$ from a to $\lambda(b)$], and consider the geodesic $\lambda(K_1)|[a, z'] \cup K_2|[z', b]$. Repeating, if necessary, the arguments used to prove relations (4) and (2), we transform this geodesic into a geodesic K_3 joining a to b and satisfying

$$K_3 \subset \sigma_1,\; K_3|[a, z_1] \subset B'(p, s/2),\; K_3|[z_1, b] \subset B'(q, s/2).$$

Similarly, rotating K_2 around a, we get a geodesic K_4 joining a and b and enjoying the properties

$$z_1 \in K_4,\; K_4 \subset \sigma_1,\; K_4|[a, z_1] \subset B'(q, s/2],\; K_4|[z_1, b] \subset B'(p, s/2),$$

$$T_1 = K_3|[a, z_1] \cup K_4|[z_1, b] \subset B'(p, s/2),\; T_2 = K_4|[a, z_1] \cup K_3|[z_1, b] \subset B'(q, s/2).$$

Let σ_2' (σ_2'') be the closed subdomain bounded by the geodesics $K_3|[a, z_1]$, $K_4|[a, z_1]$ (respectively, $K_3|[z_1, b]$, $K_4|[z_1, b]$). Then

$$\tau_1 \subset \sigma_2 = \sigma_2' \cup \sigma_2'' \subset \sigma_1.$$

Now let z_2 be any point on $\tau_1|[a, z_1)$ or $\tau_1|(z_1, b]$. Repeating the arguments above, we find geodesics K_5, K_6 which join a to b, and such that $z_1, z_2 \in K_5, K_6$; $K_5, K_6 \subset \sigma_2$, $K_5|[a, z_2] \subset B'(p, s/2)$, $K_5|[z_2, b] \subset B'(q, s/2)$, $K_6|[a, z_2] \subset B'(q, s/2)$, $K_6|[z_2, b] \subset B'(p, s/2)$ $T_3 = K_5|[a, z_2] \cup K_6|[z_2, b] \subset B'(p, s/2)$, $T_4 = K_6|[a, z_2] \cup K_5|[z_2, b] \subset B'(q, s/2)$. Moreover, if σ_3' (σ_3'') is the closet subdomain bounded by the geodesics $K_5|[a, z_2]$, $K_6|[a, z_2]$ (respectively, $K_5|[z_2, b]$, $K_6|[z_2, b]$), then

$$\tau_1 \subset \sigma_3 = \sigma_3' \cup \sigma_3'' \subset \sigma_2.$$

After n steps, we get two geodesics K_{2n-1}, K_{2n}, which join a to b, and such that $z_1, \ldots, z_{n-1} \in K_{2n-1}, K_{2n}$, where $z_1, \ldots, z_{n-1} \in \tau_1$, K_{2n-1}, $K_{2n} \subset \sigma_{n-1}$, $K_{2n-1}|[a, z_{n-1}] \subset B'(p, s/2)$, $K_{2n-1}|[z_{n-1}, b] \subset B'(q, s/2)$, $K_{2n}|[a, z_{n-1}] \subset B'(q, s/2)$, $K_{2n}|[z_{n-1}, b] \subset B'(p, s/2)$, $T_{2n-3} \equiv K_{2n-1}|[a, z_{n-1}] \cup K_{2n}|[z_{n-1}, b] \subset B'(p, s/2)$, $T_{2n-2} \equiv K_{2n}|[a, z_{n-1}] \cup K_{2n-1}|[z_{n-1}, b] \subset B'(q, s/2)$. Moreover, if σ_n' (σ_n'') is the closet subdomain bounded by the geodesics $K_{2n-1}|[a, z_{n-1}]$, $K_{2n}|[a, z_{n-1}]$ (respectively, $K_{2n-1}|[z_{n-1}, b]$, $K_{2n}|[z_{n-1}, b]$), then

$$\tau_1 \subset \sigma_n \equiv \sigma_n' \cup \sigma_n'' \subset \sigma_{n-1}.$$

The sequence $\{z_n\}$ may be chosen to be dense on τ_1.

Finally we get a sequence of geodesics $\{M_n\}_{n=1}^{\infty}$, where $M_n = T_{2n-1}$, $M_{n+1} = T_{2n}$, which converges to the arc τ_1 ([2], p. 39). Consequently, τ_1 is a geodesic. Since the spheres $S(p, s/2)$ and $S(q, s/2)$ are compact, they are geodesics in the space M.

This proves (c).

(d) One may assume, with no loss of generality, that $\tau_1 = \tau$, i.e., $\tau = S(p, s/2) \cap S(q, s/2)$ is a geodesic with extremities a and b. Moreover, assume that τ, as the common part of the spheres $S(p, s/2)$, has maximal length, and that it still remains a geodesic. To emphasize the dependence of τ upon s, we write $\tau = \tau(s)$.

Let c be the middle of geodesic τ, and let pq be a geodesic which lies in intσ, except for its extremities p, q. Let s' be arbitrary, $0 < s' < s$, and let $p_1, q_1 \in pq$ such that $p_1 \in B(p, s/2)$, $q_1 \in B(q, s/2)$, $\rho(cp_1) = \rho(cq_1) = s'/2$. Obviously, any geodesic from the collection $\{(p_1q_1)_i\}_{i\in I}$ may be obtained as the restriction to $[p_1, q_1]$ of a geodesic belonging to $\{(pq)_i\}_{i\in I}$. Since $\tau(s)$ is maximal, we see that $S(p_1, s'/2) \cap S(q_1, s'/2) \subset S(p, s/2) \cap S(q, s/2)$ and

$$s[\tau(s')] \leqslant s[\tau(s)] \quad \text{for} \quad s' \leqslant s, \tag{5}$$

where $\tau(s')$ is the common arc of the spheres $S(p_1, s'/2)$ and $S(q_1, s'/2)$.

Now let $\Delta(s)$ denote the maximal radius of a sphere with center c entirely contained in σ.

Then obviously

$$\Delta(s') \leqslant \Delta(s) \quad \text{for} \quad s' \leqslant s. \tag{6}$$

(e) If $s_1 \leq s/2$ and $s_1 + s_2 = s$, then the spheres $S(p, s_1)$ and $S(q, s_2)$ have a common simple arc $\zeta(s_1)$ with extremities L_1 and L_2 and which is contained entirely in σ, i.e.,

$$\zeta(s_1) = S(p, s_1) \cap S(q, s_2) \cap \sigma = \sigma \cap B'(p, s_1) \cap B'(q, s_2), \quad \sigma = [B'(p, s_1) \cup B'(q, s_2)] \cap \sigma.$$

The proof of (e) is similar to the proofs of formulas (1) and (2).

(f) The arc $\zeta(s_1)$ is a geodesic for $s_1 < \min(s, \Delta(s))/2$. Indeed, $\zeta(s_1)$ lies on that part of the sphere $S(q, s_2)$ which is the geodesic $\tau(2s_2)$. This is a consequence of two facts: firstly, according to (5),

$$s[\tau(2s_2)] \geqslant s[\tau(s)] \quad \text{for} \quad s_1 \leqslant s/2$$

and secondly, for $2s_1 < \Delta(s) \leq \Delta(2s_2)$ [see (6)] the arc $\zeta(s_1)$ lies precisely on the arc $\tau(2s_2)$ of $S(q, s_2)$.

(g) Further, assume that the geodesics $L_1|[p, a]$, $L_2|[p, b]$, where $a \in L_1$ and $b \in L_2$ are the extremities of arc $\tau(s)$, have the property that there are no geodesics pa, pb which contain points different from p, a, b and which lie outside σ. Clearly, this assumption does not restrict the generality of the discussion.

Let K_φ denote the sector bounded by the geodesics $\tilde{L}_1 \equiv L_1|[p, a]$ and L_φ, where $L_\varphi = \varphi(\tilde{L}_1)$, and $\varphi \in I_0(p)$ is a rotation such that $\varphi(a) \in \tau(s)|[a, c]$, c the middle of arc $\tau(s)$. Then K_φ has the following properties:

i_1) $K_\varphi \subset \sigma \cap B'(p, s/2)$;

i_2) the angle θ_φ between $\tilde{L}_1$ and L_φ at point p is different from zero;

i_3) there is φ such that $\theta_\varphi \neq 2n\pi$ ($n = \pm 1, \pm 2, \ldots$).

Indeed, i_1) is a consequence of the assumption about σ made at the beginning of (g). Suppose that i_2) does not hold for K_φ, i.e., $\theta_\varphi = 0$. According to (f), for the sector K_φ we have a sufficient criterion of the additivity of angles ([13], p. 21). This implies that the angle of sector K_φ is zero. If we now rotate K_φ around the point p using $I_0(p)$ we can get a partition of the ball $B(p, s/2)$ into a finite number [recall that the curve $S(p, s/2)$ is rectifiable] of successively adjacent sectors $K_1, \ldots, K_m$ similar to K_φ, i.e., all having angle zero. But this shows that the total angle around point p is zero, which contradicts axiom (A_5). Thus i_2) is true.

To see that one can find a sector K_φ with $\theta_\varphi \neq 2n\pi$, assume the opposite, i.e., that $\theta_\varphi = 2n\pi$ for all φ. Choose the sector K_φ with minimal θ_φ and divide it into adjacent sectors $K_{\varphi_1}, \ldots, K_{\varphi_m}$. The sum of the angles of these sectors is $\sum_i \theta_{\varphi_i} > \theta_\varphi$, which contradicts the additivity property of angles for K_φ. Consequently, i_3) is true.

(h) Let K_φ be a sector satisfying i_1)–i_3), and let $Q = K_\varphi \cup \varphi(K_\varphi)$. Then Q is a sector bounded by the geodesics $\tilde{L}_1$ and $\varphi(L_\varphi) = \varphi^2(\tilde{L}_1)$. Its angle is different from 0 and $4n\pi$. Set

$$\eta(t) = \zeta(t) \cap Q, \quad t \leqslant s_1 < \frac{1}{2} \min(s, \Delta(s)),$$

where $\zeta(t)$ is a geodesic, $\zeta(t) = \sigma \cap S(p, t)$, and let a_t, b_t and c_t be the extremities and the middle of $\eta(t)$, respectively. The geodesic L_φ is obviously the locus of the midpoints of arcs $\eta(t)$. We have $t = \rho(pa_t) = \rho(pb_t) = \rho(pc_t)$ and hence, the angles θ between $\tilde{L}_1$ and $\varphi(L_\varphi)$, satisfies

$$\cos\theta = \lim_{t\to 0}(1 - \lambda^2/2t^2), \tag{7}$$

where $\lambda = s[\eta(t)]$.

On the other hand, $\theta_\varphi = \angle(\tilde{L}_1, L_\varphi) = \angle(L_\varphi, \varphi(L_\varphi))$, and

$$\cos\theta_\varphi = \lim_{t\to 0}(1 - \lambda^2/8t^2). \tag{8}$$

The additivity of angles for Q yields

$$\theta = \angle(\hat{L}_1, L_\varphi) + \angle(L_\varphi, \varphi(L_\varphi)) = 2\theta_\varphi,$$

i.e.,

$$\theta_\varphi = \theta/2. \tag{9}$$

Now (7) and (8) imply,

$$\cos\theta = 4\cos\theta_\varphi - 3, \tag{10}$$

while from (9) we get

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 2\cos^2\theta_\varphi - 1. \tag{11}$$

Finally, from (10) and (11) we get

$$\cos^2\theta_\varphi - 2\cos\theta_\varphi + 1 = 0,$$

which in turn implies $\cos\theta_\varphi = 1$, or $\theta_\varphi = 2n\pi$ ($n = 0, \pm 1, \ldots$). But $\theta_\varphi \neq 2n\pi$. Contradiction.

This contradiction completes the proof of Theorem 3, and we see that one may take $\eta_x = \varepsilon_0$.

5. PROOF OF THEOREM 4

(a) Since M is a complete space with intrinsic metric, it is boundedly compact ([13], p. 75).

If we are given two distinct points x and z, then the completeness of M ensures the existence of a geodesic xz. Consequently, if $y \in xz$, then (xyz), and so any complete r-space satisfies axioms (I)–(III) for G-spaces.

(b) Let us prove the validity of axiom (IV). Given $x \in M$, Lemma 2 of Sec. 4 shows that there is $\varepsilon > 0$ such that the ball $B(u, \varepsilon)$ admits a rotation for all $u \in B(x, \varepsilon)$. Take $y, z \in B(x, \varepsilon/8)$. Then $\rho(yz) < \varepsilon/4$. Let $v \in S(x, \varepsilon/2)$ and since $\varepsilon < \delta_x$, there is a geodesic xv. Now let λ and φ be rotations of the spheres $B(x, \varepsilon)$ and $B(y, \varepsilon)$, respectively, such that $y \in \lambda[xv]$, $z \in \varphi[\lambda[xv]]$. Since $\rho(yz) < \varepsilon/4$ and $s(\varphi[\lambda(xv)] | [y, \lambda(v)]) > 3\varepsilon/8$, there is a point u on the geodesic $\varphi[\lambda(xv) | [y, \lambda(v)]$ such that (yzu). The choice $\rho_x = \varepsilon/8$ proves (b).

(c) M satisfies axiom (V). Indeed, let x, y, z_1, z_2 be points such that (yxz_1), (yxz_2) and $\rho(xz_1) = \rho(xz_2)$. Let us show that $z_1 = z_2$. There are geodesics yz_1 and yz_2 such that $x \in yz_1$ and $x \in yz_2$ ([2], p. 44). With no loss of generality, one can assume that $yz_1 | [x, z_1] \cap yz_2 | [x, z_2] = \{y\}$ and $yz_1 | [y, x] = yz_2 | [y, x]$. Now use part (b) from the proof of Theorem 3 and let $U = B(x, \varepsilon_0)$, as in (b). Choose $u \in yz_1 | [y, x]$ and $u \neq x$, $w_i \in yz_i | [xz_i]$ $(i = 1, 2)$, such that $\rho(ux) = \rho(xw_1) = \rho(xw_2) = \varepsilon_0/4$. Moreover, let $\varphi \in I_0(u)$ be a rotation taking w_1 into w_2, i.e., $\varphi(w_1) = w_2$. Then $L_1 = yz_2 | [u, w_2]$ and $L_2 = \varphi(yz_1 | [u, w_1])$ are geodesics joining u and w_2, and $L_1, L_2 \subset B(u, \varepsilon_0/2) \subset B(x, \varepsilon_0) = U$.

Let v be the midpoint of the geodesic $yz_1|[u, x]$, and let $\lambda \in I(v)$ be a rotation which takes u into x. Then $\lambda(L_1)$ and $\lambda(L_2)$ are two geodesics which join x with $\lambda(w_2) \in U$. However, $\lambda(L_1)$ does not coincide to $\lambda(L_2)$, because L_1 is not the same as L_2: by Proposition 1 of Sec. 3, $\varphi(y) \neq y$, while $y \in L_1$, and $\varphi(y) \in L_2$. But the existence of distinct geodesics which join x with $\lambda(w_2)$ contradicts Theorem 3, and this proves (c).

(d) Consequently, M is a Buseman G-space. Now we see that Theorem 4 is a result of Theorem B [2]. This completes the proof of Theorem 4.

Proof of the Corollary. The two-dimensional locally Euclidean G-spaces are described in [2]. They are: Euclidean plane, cylinder, Möbius band, torus and Klein bottle. The only two-dimensional locally spherical G-spaces are the sphere and the projective plane [2]. The description of the two-dimensional locally hyperbolic spaces is given in [7] (see also [2]).

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PROOF OF THE VAN DER WAERDEN CONJECTURE FOR PERMANENTS

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1°. We prove (Theorem 1) the validity of the van der Waerden conjecture, formulated by him in 1926 ([1; 2, p. 155, Conjecture 1]), regarding the minimum of the permanent of a double stochastic matrix. In the course of the proof one answers positively (Theorem 2) the Marcus–Newman conjecture on the permanent of a doubly stochastic matrix ([2], p. 156, Conjecture 11; [3], Conjecture 11). The proof of Theorem 2, and with it also that of Theorem 1, is based on the representation of the permanent in terms of mixed discriminants and on the subsequent use of a geometric inequality for the permanents (Lemma), which follows directly from Aleksandrov's known inequalities for mixed discriminants [4]. The reduction from Theorem 2 to Theorem 1 is known and is based on the results of [5, 6]. As a consequence of Theorem 1 we obtain lower estimates, formulated previously by other authors (see [2], Sec. 8.2; 7, 8) under the assumption of the validity of the van der Waerden conjecture and improving in an essential manner the known estimates for the number of Latin rectangles and squares, the number of nonisomorphic Steiner triple systems and for the key constant λ_d in the d-dimensional dimer problem. We indicate some other applications of the results obtained in the paper.

2°. By the permanent of an $n \times n$ matrix $A = (a_{ij})$ over the field of complex numbers we mean the expression (see, for example, [9])

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