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ON THE FOUNDATIONS OF SPACE-TIME GEOMETRY

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We consider affine n-dimensional space A^n, $n \geqslant 0$, with a given disconnected order, invariant with respect to all parallel translations.

Geometrically, the introduction of an *order* in A^n consists in associating with each point $x \in A^n$ a set $P_x \subset A^n$ with the following conditions:

1) $x \in P_x$;

2) if $y \in P_x$ then $P_y \subset P_x$;

3) $P_x \neq P_y$ if $x \neq y$.

Then, writing the relation $y \in P_x$ as $x \leqslant y$, we obtain an *order* (partial) in A^n.

The invariance of the order with respect to parallel translations is understood in the following manner. If t is a parallel translation and $t(P_x)$ is the image of the set P_x under the translation t, then $t(P_x) = P_t(x)$ for any point $x \in A^n$ and any translation t.

Thus an invariant order is defined by a set P_e, related to a fixed point e.

The order P_e is called *disconnected* if the set P_e is not connected; otherwise the order is *connected.*

In this paper we present some results about one-to-one mappings of A^n onto itself which preserve the disconnected order in A^n. In this connection we say that a one-to-one mapping $f\colon A^n \longrightarrow A^n$ onto A^n *preserves the order* P_e, or is *P-isotone,* if $f(P_x) = P_{f(x)}$ for any point $x \in A^n$. It is easy to verify that P-isotonicity is equivalent to the monotonicity of f and f^{-1}, i.e. the relation $x \leqslant y$ implies $f(x) \leqslant f(y)$ and $f^{-1}(x) \leqslant f^{-1}(y)$.

1. We shall fix the point e through the whole paper and write P instead of P_e. If M is any set in A^n containing e then M_x denotes the set obtained from M by a translation t such that $t(e) = x$. We denote by int A, $\overline{A}$ and ∂A the interior, the closure and the boundary of A, respectively. Moreover, $L(x, y)$ denotes the ray going out from the point x through the point y, $x \neq y$; $x \in L(x, y)$.

DEFINITION 1. A *shift* $d_{E\mathbf{l}}$ (or d_{EL}), where E is a hyperplane and $\mathbf{l}$ is a vector (L is a ray) not parallel to E, is a homeomorphism from A^n onto itself that satisfies the following conditions:

(a) On each hyperplane E_a parallel to E, $d_{E\mathbf{l}}$ (d_{EL} respectively) is a translation.

(b) $d_{E\mathbf{l}}$ (or d_{EL}) takes segments (rays) that are equal and parallel to $\mathbf{l}$ (L) into segments (rays) that are equal and parallel to $\mathbf{l}$ (L).

DEFINITION 2. A *quasicylinder* $Q(E, \mathbf{l})$ is a set M which satisfies the following conditions [1]:

(a) There are hyperplanes $E_1, E_2, \ldots$, which are parallel to E, where E_{i+1} is obtained from E_i by the translation corresponding to the vector $\mathbf{l}$ such that the set M can be

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represented as

(1) $M = \bigcup_i [M_i \cup (M \cap E_i)]$,

where each M_i is a cylinder formed by open segments equal to **1** (as vectors) with their ends on E_i and E_{i+1} (the case when some or even all of the M_i are empty is not excluded).

(b) M does not admit a representation (1) with the same hyperplane E and a vector $\mathbf{1}'$ parallel to **1**, but not equal to **1**.

Intuitively, a quasicylinder consists of cylinders with equal and parallel generators placed one on another; the bases of the cylinders are separated and between the cylinders there is a padding in the hyperplane E_i.

A quasicylinder can be characterized as a set M such that there exist a hyperplane E, a vector **1** and a translation t with $d_{E\mathbf{1}}(M) = t(M)$ ([1], §6.2).

The definition of a quasicylinder $Q(E, L)$, where L is a ray, is analogous.

2. The disconnected order P which we are considering satisfies the following conditions:

(A) $P = \{e\} \cup Q$, where Q is a closed connected set with interior points and does not contain e.

(B) P is inside of a convex cone with an acute vertex e (*acute vertex* means that the cone does not contain any straight line).

The cone

(2) $C = \overline{\bigcup_{x \in \operatorname{int} Q} L(e, x)}$

with the vertex e is called an *exterior cone.*

DEFINITION 3. An order P is said to be *ruled* if there exists a ray $L(e, x_0) \subset C$, where C is the exterior cone (2), such that for any straight line λ parallel to $L(e, x_0)$ the set $\lambda \cap Q$ is either empty, or is necessarily a ray.

DEFINITION 4. An order P is said to be *m-ruled,* where $m = 1, 2, \ldots$ is a natural number, if P is ruled with respect to the rays $L_1(e, x_1), \ldots, L_m(e, x_m)$, which are not on the same $(m-1)$-dimensional plane.

If an order P is not ruled, it is called *non ruled.*

THEOREM 1. *Let P be a disconnected n-ruled order in A^n, $n \geqslant 2$, which satisfies conditions* (A) *and* (B). *Then either any P-isotone one-to-one mapping f: $A^n \longrightarrow A^n$ onto A^n is an affine mapping, or P is a quasicylinder and f is of the form*

(3) $f = f_0 \circ d_1 \circ \ldots \circ d_p$,

where f_0 is an affine mapping and the d_i are either $d_{E_i\mathbf{1}_i}$ or $d_{E_iL_i}$ (the order of the d_i in (3) *is not essential).*

THEOREM 2. *Let P be a disconnected non ruled order in A^n, $n \geqslant 2$, which satisfies conditions* (A) *and* (B). *Then there exists a connected order, given by an m-dimensional cone $K \subset C$, $m \geqslant 1$, with the vertex e, such that any P-isotone one-to-one mapping onto is K-isotone.*

COROLLARY 1. *There exists a ruled order $\widetilde{P} = \{e\} \cup \widetilde{Q}$ such that:*

1) *$Q \subset \widetilde{Q}$ and $\widetilde{P}$ satisfies conditions* (A) *and* (B); *and*

2) *any P-isotone one-to-one mapping onto is $\widetilde{P}$-isotone.*

Thus one need consider only ruled orders. Note that neither in Theorem 2 nor in Corollary 1 is it claimed that the order is n-ruled, but only that it is ruled.

The concept of a ruled order is not artificial or technical; it has a very deep basis. As follows from Theorem 3 below, the ruled property is a consequence of homogeneity of the boundary ∂Q of the order P.

Let G_a be the set of all P-isotone one-to-one mappings onto such that $g(a) = a$ for any $g \in G_a$. Then we have condition G1: The group G_a acts transitively on ∂Q_a, i.e. for any $x, y \in \partial Q_a$ there exists a $g \in G_a$ such that $g(x) = y$.

THEOREM 3. *Any disconnected order P in A^n, $n \geqslant 2$, which satisfies conditions* (A), (B) *and* G1 *is at least* 1*-ruled.*

3. We introduce the following condition.

G2. There is no m-dimensional plane, $1 \leqslant m < n$, through the point e which can be mapped onto itself by any isotone mapping $g \in G_e$.

THEOREM 4. *Let P be a disconnected order in A^n, $n \geqslant 2$, which satisfies* (A), (B) *and* G2. *Then the assertion of Theorem* 1 *holds.*

THEOREM 5. *Let $P = \{e\} \cup Q$ be a disconnected order in A^n, $n \geqslant 2$, which satisfies conditions* (A), (B) *and* G2.

Then P defines an order with the condition G1 *if and only if Q is a set derived from a cone K with an affine group, transitive within K, by planes which cut a constant volume from K. In this connection G_e is an unimodular subgroup of this affine group. Condition* G2 *can be replaced by the requirement that the order P be n-ruled.*

This theorem follows from Theorem 1 and Theorem 5 in A. D. Aleksandrov's paper [2].

It is easy to see that $K = C$, i.e. K coincides with the exterior cone.

COROLLARY 2. *Let the assumption of Theorem 5 hold.*

Then in the 4-dimensional space A^4 the exterior cone is either elliptical (and G_e is the Lorentz group), or a quadrihedral angle, or the cartesian product of a ray and a 3-dimensional elliptical cone.

This follows from Theorem 5 and Theorem 8 in [3].

4. The following statements show that the assumption that Q is a connected set and contains interior points can be omitted.

PROPOSITION 1. *Let P be an order with respect to the ray L, which satisfies conditions* (A) *and* (B) *except that Q is not connected.*

Then there exists an order $\widetilde{P}$ such that

1) *$\widetilde{P}$ is lined with respect to the ray L;*

2) *$\widetilde{P}$ satisfies conditions* (A) *and* (B); *and*

3) *any P-isotone one-to-one mapping onto is $\widetilde{P}$-isotone.*

PROPOSITION 2. *Let P be a disconnected order, which satisfies conditions* (B) *and* G1. *Let $e \notin \overline{Q}$. Then Q contains interior points.*

5. Now we formulate a result which is very important to the foundations of the special theory of relativity.

THEOREM 6. *Let P be a disconnected order in A^4 which satisfies conditions* (A), (B), G1 *and* G2. *Assume that the exterior cone C is not a quasicylinder or a quadrihedral angle. Then*

1) *C is a closed elliptic cone;*

2) *G_e is the homogeneous Lorentz group; and*

3) *there exists a cartesian coordinate system x_0, x_1, x_2, x_3 such that ∂Q is given by the relation $x_0^2 - x_1^2 - x_2^2 - x_3^2 = m^2, x_0 > 0$, where $m = \text{const} \neq 0$, and the cone ∂C is given by $x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0, x_0 \geqslant 0$.*

Condition G2 *can be replaced by the assumption that P is a 4-lined order.*

6. **Physical interpretation.** The affine space A^4 is considered as the state space, or, what is the same, space-time. To each point of this space a state is related; x and y denote points and states, respectively. The relation $x \leqslant y$ shows that x acts on y, i.e. an energy impluse is transmitted from x to y. Condition (B) is a restriction on the velocity of the transmission of the action. In this case (A) says that the action is transmitted by a jump that omits the states in a sufficiently small space-time domain. The requirement of isotopy and homogeneity of the space is given by conditions G1 and G2.

Theorem 6 is a system of axioms which define the geometry of a Minkowski space. Therefore we can draw a very important conclusion. One can verify that the geometry of space-time is pseudo-euclidean without assuming the existence of cause-and-effect relations in the domain of microphenomena. Thus the Lorentz group is a consequence of causal relations of the microcosmos, and the structure of the microcosmos is to some extent the result of this fundamental symmetry of space-time.

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