CHRONOGEOMETRY OF GÖDEL AND DE SITTER MANIFOLDS

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We consider a four-dimensional, elementary (i.e., diffeomorphic to Euclidean space R^4) Lorentz manifold V^4. On V^4 it is possible to introduce coordinates x^0, x^1, x^2, x^3 in which the Lorentz metric g is given by the differential form

$$ds^2 = \sum_{i,k=0}^{3} g_{ik}(x^0, x^1, x^2, x^3)\, dx^i dx^k.$$

An isotropic cone $\tilde{C}_{x_0}$ at the point $x_0 \in V^4$ is a cone lying in the tangent space $V^4_{x_0}$ to V^4 at the point x_0, each vector ξ of which satisfies the equation

$$g_{x_0}(\xi, \xi) = 0,$$

or in coordinates:

$$\sum_{i,k=0}^{3} g_{ik}(x_0^0, x_0^1, x_0^2, x_0^3)\, \xi^i \xi^k = 0,$$

where $x_0 = (x_0^0, x_0^1, x_0^2, x_0^3)$.

To the cone $\tilde{C}_{x_0}$ we assign a subset C_{x_0} of the manifold V^4 as follows: a point $x \in C_{x_0}$ if and only if it satisfies the equation

$$\sum_{i,k=0}^{3} g_{ik}(x_0^0, x_0^1, x_0^2, x_0^3)\,(x^i - x_0^i)(x^k - x_0^k) = 0.$$

If x^0, x^1, x^2, x^3 are considered as affine coordinates, then C_x is a cone in V^4. Let $f: V^4 \to V^4$ be an arbitrary differentiable mapping. We say that f preserves the family $\{C_x : x \in V^4\}$ if the following condition is satisfied:

$$f(C_x) = C_{f(x)}. \tag{1}$$

It is not hard to verify that in this case the differential df of the mapping f preserves the isotropic cones $\tilde{C}_x$, i.e.,

$$(df)_x(\tilde{C}_x) = \tilde{C}_{f(x)}. \tag{2}$$

The converse assertion is, in general, not true. However, if in the coordinates x^0, x^1, x^2, x^3 f is given by an affine transformation (is affinizable), then Eq. (2) implies Eq. (1).

It is easy to demonstrate the validity of the following assertion.

LEMMA. If G_r is an r-parameter group of motions of the manifold (V^4, g) which is affinitely representable in the coordinates x^0, x^1, x^2, x^3 (i.e., each motion $\varphi \in G_r$ is affinizable), then Eqs. (1) and (2) are equivalent for any motion $\varphi \in G_r$.

A standard problem of chronogeometry consists in determining mappings $f: V^4 \to V^4$ satisfying a condition of the form (1). Differentiability of f is not required, and the condition of continuity is often given up as well. Knowledge of the group of motions of a Lorentz manifold makes it possible to establish its geometry. The converse problem is also interesting: find a group of motions without assuming the differentiability of its transformations (the usual approach leading to the concept of the Killing vector) and dealing only with isotropic cones or, more precisely, with the "cones" C_x; i.e., the following question is posed: is it possible to construct the group of motions G_r of a manifold V^4 assuming that each transformation $\varphi \in G_r$ satisfies condition (1)?

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The lemma shows that this approach is justified at least for groups of notions admitting affine representation. For example, solvable groups of motions G_3 acting transitively on V^3 (see [1]) are such groups.

Definition. Suppose that a group of motions G_r is affinizable in the coordinates $\{x^i\}$. In this case a mapping preserving isotropic cones is understood to be a bijection $f: V^4 \to V^4$ satisfying condition (1).

THEOREM A (Aleksandrov and Ovchinnikova [2]). Any mapping preserving isotropic cones in a Minkowski world

$$ds^2 = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2}$$

is a superposition of an inhomogeneous Lorentz transformation (i.e., a motion) and a similarity transformation.

We formulate the main results of the paper.

THEOREM 1. Any homeomorphic mapping of the Gödel universe [3, 4]

$$ds^2 = a^2\left(dx^{0^2} - dx^{1^2} - \frac{1}{2} e^{2x^1} dx^{2^2} - dx^{3^2} + 2e^{x^1} dx^0 dx^2\right),$$

where $a = \text{const} \neq 0$, which preserves isotropic cones is a motion.

The group of motions of the Gödel manifold has type G_4VI_1 in the classification of Petrov [1] and can be represented by affine transformations of the form

$$\bar{x}_0 = x^0 + \alpha;\ \bar{x}^1 = x_1 + \beta;\ \bar{x}^2 = x^2 e^{-\beta} + \gamma;\ \bar{x}^3 = x^3 + \delta, \tag{3}$$

where α, β, γ, δ are parameters. This group acts simply and transitively on V^4.

THEOREM 2. Any homeomorphic mapping of the de Sitter universe [3]

$$ds^2 = dx^{0^2} - e^{kx^0}\left(dx^{1^2} + dx^{2^2} + dx^{3^2}\right),$$

where $k = \text{const}$, which preserves isotropic cones is a motion [5].

The group of motions of the de Sitter universe G_7 in the coordinates $\{x^i\}$ cannot be represented as a subgroup of the group of affine transformations.

In the coordinates

$$y^0 = \exp(-kx^0),\ y^\alpha = kx^\alpha\ \ (\alpha = 1,\ 2,\ 3) \tag{4}$$

the metric of the de Sitter universe has the following form:

$$ds^2 = \left(\frac{1}{ky^0}\right)^2 \left(dy^{0^2} - dy^{1^2} - dy^{2^2} - dy^{3^2}\right), \tag{5}$$

and the group G_7 consists of transformations of the form

$$f(y) = \lambda \begin{pmatrix} 1, 0 \ldots 0 \\ 0 & \\ \vdots & \boxed{U} \\ 0 & \end{pmatrix} y + \begin{pmatrix} 0 \\ \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad \lambda > 0,$$

where U is an orthogonal matrix, i.e., G_7 becomes affine in the coordinates $\{y^i\}$.

The group G_7 contains a simply transitive subgroup G_4VI_1 which becomes affine in the coordinates (4). Therefore, the cones C_x in the Gödel and de Sitter universes are obtained by "transport" of the cone C_{x_0}, where x_0 is a fixed point, by means of the noncommutative group G_4VI_1 in contrast to the Minkowski world, where "transport" is realized by a commutative group of translations.

The geometry of the Minkowski world is flat, the de Sitter universe has constant curvature, and the Gödel universe is essentially curved.

From the point of view of the theory of relativity Theorems A, 1, and 2 bespeak the fact that the Minkowski, Gödel, and de Sitter geometries can be determined by knowledge only of the law for the propagation of light.

Let

$$K_x^+ = \left\{ u \in V^4 : u^0 > 0, \quad \sum_{i,k=0}^{3} g_{ik}(x)(u^i - x^i)(u^k - x^k) > 0 \right\} \cup \{x\},$$

$$K_x^- = \left\{ u \in V^4 : u^0 < 0, \quad \sum_{i,k=0}^{3} g_{ik}(x)(u^i - x^i)(u^k - x^k) > 0 \right\} \cup \{x\}.$$

Then the following result holds.

THEOREM 3. The sets of the form $K_x^+ \cap K_y^-$, where $x \in K_y^-$ and x, y are arbitrary points, form a basis for the topology of the Gödel and de Sitter universes.

The proof of Theorem 3 is essentially no different than the proof of the analogous result for a Minkowski world [6]. In the case of the de Sitter universe it is necessary to go over to the coordinates (4).

The family of sets $\{K_x^+ : x \in V^4\}$ in the case of the de Sitter universe defines an ordering in the sense of the paper [7]. In [7] it is shown that any mapping preserving this ordering is a motion. There is no analogous ordering in the Gödel universe.

1. Proof of Theorem 1. The cone C_Z is given by the equation

$$\sum_{i,k=0}^{3} g_{ik}(z)(x^i - z^i)(x^k - z^k) = 0$$

or

$$(x^0 - z^0)^2 - (x^1 - z^1)^2 - \frac{1}{2} e^{2z^1}(x^2 - z^2)^2 - (x^3 - z^3)^2 + 2e^{z^1}(x^0 - z^0)(x^2 - z^2) = 0, \tag{6}$$

where (z^0, z^1, z^2, z^3) are the coordinates of the point z.

We further identify V^4 with R^4. We denote by $H^2(a)$ the hyperplane in R^4 defined by the equation $x^2 = a =$ const. In each hyperplane $H^2(z_0^2)$ we have the family of cones

$$S(z^0, z^1, z^3) = \{(x^0, x^1, z_0^2, x^3) \in R^4 : (x^0 - z^0)^2 - (x^1 - z^1)^2 - (x^3 - z^3)^2 = 0\},$$

where $z \in H^2(z_0^2)$. In the cone $S(z_0^0, z_0^1, z_0^3)$, $z_0 \in H^2(z_0^2)$, we can choose four lines in general position (i.e., no three lines lie in the same two-dimensional plane) $l_{z_0}^A$ (A = 1, 2, 3, 4). Let $z \in H^2(z_0^2)$ and let t be a translation such that $t(z_0) = z$. We set $l_z^A = t(l_{z_0}^A)$ (A = 1, 2, 3, 4). In $H^2(z_0^2)$ we have four families of parallel lines $\{l_z^A\}$ (A = 1, 2, 3, 4). By Eq. (1) and

$$S(u^0, u^1, u^3) \cap S(z^0, z^1, z^3) = l_u^A = l_z^A = C_u \cap C_z, \quad z \in l_u^A,$$

we see that $f(l_z^A)$ is a line in R^4. Further, it is easy to verify that the parallel lines l_z^A and l_u^A ($z \neq u$) are mapped onto parallel lines, and the two-dimensional plane spanned by each pair of lines $\{l_z^A, l_z^B\}$, $A \neq B$, is mapped onto a two-dimensional plane. This implies that the lines $f(l_z^A)$ (A = 1, 2, 3, 4) are in general position and $f(H^2(z_0^2))$ is a hyperplane (see [2]).

Let $D_z = \bigcup_{A=1}^{4} l_z^A$, $D'_{f(z)} = \bigcup_{A=1}^{4} f(l_z^A)$. Then $f(D_Z) = D'_{f(Z)}$, and hence f is affine on $H^2(z_0^2)$ by Theorem 1 of [8].

We now consider the two-dimensional plane

$$P(ab) = \{x \in R^4 : x^1 = a, \ x^3 = b\},$$

where a, b are arbitrary constants. On the plane $P(z_0^1 z_0^3)$ we have the family of 1-cones

$$F(z^0, z^2) = \left\{(x^0, z_0^1, x^2, z_0^3) \in R^4 : (x^0 - z^0)^2 - \frac{1}{2} e^{2z^1}(x^2 - z^2)^2 + 2e^{z^1}(x^0 - z^0)(x^2 - z^2) = 0\right\}, \quad z \in P(z_0^1 z_0^3).$$

This is a pair of lines intersecting at the point (z^0, z_0^1, z^2, z_0^3) and lying in $P(z_0^1 z_0^3)$. We denote these lines by l_Z^5, l_Z^6, where $z = (z^0, z_0^1, z^2, z_0^3) \in P(z_0^1 z_0^3)$. In the plane $P(z_0^1 z_0^3)$ we obtain two families of parallel lines. Since $l_z^B = l_u^B = F(z^0, z^2) \cap F(u^0, u^2) \subset C_u \cap C_z$, $z \in l_u^B$ (B = 5,6), it follows that $f(l_Z^B)$ (B = 5, 6) is a line. Hence, $f[P(z_0^1 z_0^3)]$ is a two-dimensional plane. The plane $P(z_0^1 z_0^3)$ intersects the hyperplane $H^2(a)$ along the line $l^7_{(z_0^0, z_0^1, a, z_0^3)}$. Since for any a the image $f(H^2(a))$ is obviously a hyperplane, and $f(H^2(a))$ is parallel to $f(H^2(a'))$ ($a \neq a'$), on the plane $P(z_0^1 z_0^3)$ we obtain a third family of parallel lines $\{l_z^7 : z \in P(z_0^1 z_0^3)\}$. Thus, f maps $P(z_0^1 z_0^3)$ onto a plane and the three families of parallel lines $\{l_z^A : z \in P(z_0^1 z_0^3)\}$ (A = 5, 6, 7) onto an analogous family. This means that f is affine on $P(z_0^1 z_0^3)$ [8].

Let $z_0 \in R^4$, and let $L_{z_0}^A$ (A = 1, 2, 3, 4) be distinct lines such that $L_{z_0}^B \subset H^2(z_0^2)$ $(B = 1, 2, 3)$, $L_{z_0}^3, L_{z_0}^4 \subset P(z_0^1 z_0^3)$. There exists an affine transformation g of R^4 onto R^4 possessing the following properties:

$$g(f(z_0)) = z_0, \quad g\left(f\left(L_{z_0}^A\right)\right) = L_{z_0}^A \quad (A = 1, 2, 3, 4).$$

Then the mapping $g \circ f$ maps $H^2(z_0^2)$ affinely onto $H^2(z_0^2)$ and $P(z_0^1 z_0^3)$ affinely onto $P(z_0^1 z_0^3)$, and $(g \circ f)(L_{z_0}^A) = L_{z_0}^A$ (A = 1, 2, 3, 4). By choosing the lines $L_{z_0}^A$ (A = 1, 2, 3, 4) as new coordinate axes, we see that in these coordinates $g \circ f$ is given by an affine transformation. Hence, f affinely maps R^4 onto R^4, i.e.,

$$f^i(x) = \sum_{k=0}^{3} a_k^i x^k + a^i \qquad (i = 0, 1, 2, 3). \tag{7}$$

Since there is the equality

$$f(C_z) = C_{f(z)},$$

we must also have in addition to (6)

$$[f^0(x) - f^0(z)]^2 - [f^1(x) - f^1(z)]^2 - \frac{1}{2} e^{2f^1(z)} [f^2(x) - f^2(z)]^2 - [f^3(x) - f^3(z)]^2 + \\ + 2e^{f^1(z)} \left[f^0(x) - f^0(z)\right] \left[f^2(x) - f^2(z)\right] = 0. \tag{8}$$

From (7) and (8) we obtain

$$\left[(a_0^0)^2 - (a_0^1)^2 - \frac{(a_0^2)^2}{2} \exp\left(2\sum_{k=0}^{3} a_k^1 z^k + 2a^1\right) - (a_0^3)^2 + 2a_0^0 a_0^2 \exp\left(\sum_{k=0}^{3} a_k^1 z^k + a^1\right)\right](x^0 - z^0)^2 -$$

$$-\left[-(a_1^0)^2 + (a_1^1)^2 + \frac{(a_1^2)^2}{2} \exp\left(2\sum_{k=0}^{3} a_k^1 z^k + 2a^1\right) + (a_1^3)^2 - 2a_1^0 a_1^2 \exp\left(\sum_{k=0}^{3} a_k^1 z^k + a^1\right)\right](x^1 - z^1)^2 -$$

$$-\left[-(a_2^0)^2 + (a_2^1)^2 + \frac{(a_2^2)^2}{2} \exp\left(2\sum_{k=0}^{3} a_k^1 z^k + 2a^1\right) + (a_2^3)^2 - 2a_2^0 a_2^2 \exp\left(\sum_{k=0}^{3} a_k^1 z^k + a^1\right)\right](x^2 - z^2)^2 -$$

$$-\left[-(a_3^0)^2 + (a_3^1)^2 + \frac{(a_3^2)^2}{2} \exp\left(2\sum_{k=0}^{3} a_k^1 z^k + 2a^1\right) + (a_3^3)^2 - 2a_3^0 a_3^2 \exp\left(\sum_{k=0}^{3} a_k^1 z^k + a^1\right)\right](x^3 - z^3)^2 +$$

$$+\left[2a_0^0 a_2^0 - 2a_0^1 a_2^1 - a_0^2 a_2^2 \exp\left(2\sum_{k=0}^{3} a_k^1 z^k + 2a^1\right) - 2a_0^3 a_2^3 + \right.$$

$$\left. + 2(a_0^0 a_2^2 + a_2^0 a_0^2) \exp\left(\sum_{k=0}^{3} a_k^1 z^k + a^1\right)\right](x^0 - z^0)(x^2 - z^2) +$$

$$+\left[2a_0^0 a_1^0 - 2a_0^1 a_1^1 - a_0^2 a_1^2 \exp\left(2\sum_{k=0}^{3} a_k^1 z^k + 2a^1\right) - 2a_0^3 a_1^3 + \right.$$

$$\left. + 2(a_0^0 a_1^2 + a_1^0 a_0^2) \exp\left(\sum_{k=0}^{3} a_k^1 z^k + a^1\right)\right](x^0 - z^0)(x^1 - z^1) +$$

$$+\left[2a_0^0 a_3^0 - 2a_0^1 a_3^1 - a_0^2 a_3^2 \exp\left(2\sum_{k=0}^{3} a_k^1 z^k + 2a^1\right) - 2a_0^3 a_3^3 + \right.$$

$$\left. + 2(a_0^0 a_3^2 + a_3^0 a_0^2) \exp\left(\sum_{k=0}^{3} a_k^1 z^k + a^1\right)\right](x^0 - z^0)(x^3 - z^3) +$$

$$+\left[2a_1^0 a_2^0 - 2a_1^1 a_2^1 - a_1^2 a_2^2 \exp\left(2\sum_{k=0}^{3} a_k^1 z^k + 2a^1\right) - 2a_1^3 a_2^3 + \right.$$

$$\left. + 2(a_1^0 a_2^2 + a_3^0 a_1^2) \exp\left(\sum_{k=0}^{3} a_k^1 z^k + a^1\right)\right](x^1 - z^1)(x^2 - z^2) +$$

$$+\left[2a_1^0 a_3^0 - 2a_1^1 a_3^1 - a_1^2 a_3^2 \exp\left(2\sum_{k=0}^{3} a_k^1 z^k + 2a^1\right) - 2a_1^3 a_3^3 + \right.$$

$$\left. + 2(a_1^0 a_3^2 + a_3^0 a_1^2) \exp\left(\sum_{k=0}^{3} a_k^1 z^k + a^1\right)\right](x^1 - z^1)(x^3 - z^3) +$$

$$+\left[2a_2^0a_3^0 - 2a_2^1a_3^1 - a_2^2a_3^2\exp\left(2\sum_{k=0}^{3}a_k^1z^k + 2a^1\right) - 2a_2^3a_3^3 + \right.$$
$$\left. + 2\left(a_2^0a_3^2 + a_3^0a_2^2\right)\exp\left(\sum_{k=0}^{3}a_k^1z^k + a^1\right)\right](x^2 - z^2)(x^3 - z^3) = 0 \tag{9}$$

Comparing (9) and (6), we obtain*

$$a_0^1 = a_2^1 = a_3^1 = 0, \quad a_0^2 = a_1^2 = a_2^0 = a_3^2 = 0, \tag{10}$$
$$a_3^0 = a_1^0 = a_2^3a_3^3 = a_1^3a_3^3 = a_1^3a_2^3 = a_0^3a_3^3 = 0,$$
$$(a_0^0)^2 - (a_0^3)^2 = 1, \quad (a_1^1)^2 + (a_1^3)^2 = 1, \quad (a_3^3)^2 = 1. \tag{11}$$

From (10) and (11) we obtain

$$a_1^3 = a_2^3 = a_0^3 = 0.$$

Hence

$$(a_0^0)^2 = (a_1^1)^2 = (a_3^3)^2 = 1,$$
$$(a_2^2)^2\exp 2a^1 = 1,$$
$$a_0^0a_2^2\exp a^1 = 1.$$

It is not hard to see that $a_1^1 = 1$. Leaving aside the mappings $x^A \to -x^A$ (A = 0, 2, 3), we obtain

$$a_0^0 = a_1^1 = a_3^3 = 1, \ a_2^2 = e^{-a^1},$$

i.e., we have obtained a transformation of the form (3). This means that f is a motion. The proof of Theorem 1 is complete.

2. Proof of Theorem 2. In the coordinates (4) the metric of the de Sitter universe can be written in the form (5). Hence, V^4 is identified with the half space $\{y \in R^4 : y^0 > 0\}$, and the cones C_z are given by the equation

$$(y^0 - z^0)^2 - (y^1 - z^1)^2 - (y^2 - z^2)^2 - (y^3 - z^3)^2 = 0.$$

In [9, Theorem 2] it is shown that in this case a mapping preserving isotropic cones can be written in the form

$$f(y) = \lambda\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \boxed{U} & \\ 0 & & & \end{pmatrix} y + \begin{pmatrix} 0 \\ \alpha \\ \beta \\ \gamma \end{pmatrix},$$

where U is an orthogonal matrix, and $\lambda > 0$, α, β, γ are parameters.

All these transformations form a 7-parameter group G_7 which is easily seen to be the group of motions of the de Sitter universe. The proof of Theorem 2 is complete.

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*Conformal transformations of the type (7) are all trivial, i.e., they reduce to motions.