# NEW SOLUTION OF THE EINSTEIN – DIRAC EQUATIONS

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A new solution of the Einstein – Dirac equations is found, describing the dynamics of a self-gravitating neutrino field. The gravitational field is wavelike.

Within the framework of the theory of general relativity, the dynamics of self-gravitating spinor material are described by the Einstein – Dirac system of equations ([1], p. 142):

$$R_{i\kappa} - \frac{1}{2} g_{i\kappa} R = \frac{8\pi G}{c^4} T_{i\kappa}, \tag{1}$$

$$i\hbar\gamma^{\kappa}\left(\frac{\partial\psi}{\partial x^{\kappa}} - \Gamma_{\kappa}\psi\right) - mc\psi = 0, \tag{2}$$

where $\psi$ is a bispinor,

$$T_{i\kappa} = \frac{i\hbar c}{4}\left\{\psi^{*}\gamma^{(0)}\gamma_{i}\left(\frac{\partial\psi}{\partial x^{\kappa}} - \Gamma_{\kappa}\psi\right) - \left(\frac{\partial\psi^{*}}{\partial x^{\kappa}}\gamma^{(0)} + \psi^{*}\gamma^{(0)}\Gamma_{\kappa}\right)\gamma_{i}\psi + \psi^{*}\gamma^{(0)}\gamma_{\kappa}\left(\frac{\partial\psi}{\partial x^{i}} - \Gamma_{i}\psi\right) - \left(\frac{\partial\psi^{*}}{\partial x^{i}}\gamma^{(0)} + \psi^{*}\gamma^{(0)}\Gamma_{i}\right)\gamma_{\kappa}\psi\right\} -$$

is the energy – momentum tensor of the spinor field (the superscript * on $\psi$ denotes the Hermitian conjugate value) ([2], p. 381).

Further

$$\Gamma_{\kappa} = \frac{1}{4} g_{ml}\left(\frac{\partial\lambda_{r}^{(s)}}{\partial x^{\kappa}}\lambda_{(s)}^{l} - \Gamma_{r\kappa}^{l}\right)s^{mr};$$
$$s^{mr} = \frac{1}{2}(\gamma^{m}\gamma^{r} - \gamma^{r}\gamma^{m}); \quad \gamma^{\kappa} \equiv \lambda_{(i)}^{\kappa}\gamma^{(i)}, \tag{4}$$

where $\lambda_{(i)}^{K}$ is the i-th tetrad vector and $\gamma^{(i)}$ is the Dirac matrix.

The indices denoted by Latin letters take on values 0, 1, 2, 3, while the Greek letter indices take on values 1, 2, 3.

Study of solutions of the Einstein – Dirac equations will in time provide answers to the following questions: 1) whether a broader concept of the structure of matter, using the concept of internal degrees of freedom (spin) of the medium, will change the problem of singularities ([1], p. 133); 2) whether geometrization of neutrine fields along the lines of Wheeler's and Rainich's neutrino is possible.

Thus the derivation of particular solutions of the Einstein – Dirac equations takes on additional importance. As is well known, the solution of Eqs. (1), (2) is an extremely complex problem, strongly dependent on successful choice of the metric $g_{ik}$. It should be understood that the metric $g_{ik}$ defines the geometry of space – time and cannot be selected arbitrarily; it is necessary that $g_{ik}$ correspond to a definite physical problem, i.e., that it admit a physical interpretation [1, 3, 4, 5].

## 1. GRAVITATIONAL RADIATION OF A NEUTRINO FLUX

We will consider only neutrino fields, i.e., in Eq. (2) we take m = 0. We assume that the neutrino field is planar and propagates in the positive x direction, creating a wavelike gravitational field. Since in this case the metric $g_{ik}$ must describe a field of planar gravi-

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tational waves, also propagating along the x-axis, the metric can conveniently be taken in the form ([6], p. 292):

$$ds^2 = 2dx^0 dx^1 - \exp[2P(x^0)]\,dx^{2^2} - \exp[2Q(x^0)]\,dx^{3^2}, \tag{5}$$

where the coordinates $x^0$, $x^1$, $x^2$, $x^3$ are related to the coordinates x, y, z, and t by the relationships

$$x^0 = \frac{1}{\sqrt{2}}(ct - x); \quad x^1 = \frac{1}{\sqrt{2}}(ct + x); \quad x^2 = y; \quad x^3 = z.$$

We choose the following tetrad:

$$\lambda^i_{(0)} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right); \quad \lambda^i_{(1)} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right); \tag{6}$$

$$\lambda^i_{(2)} = (0, 0, e^{-P}, 0); \quad \lambda^i_{(3)} = (0, 0, 0, e^{-Q})$$

and $\gamma$, the Dirac matrix

$$\gamma^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^{(\alpha)} = \begin{bmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{bmatrix},$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We assume that $\psi$ depends solely on $x^0$ and

$$\psi = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

Performing calculations with Eqs. (3), (4), we find (primes denote differentiation with respect to $x^0$):

$$\Gamma_0 = \Gamma_1 = 0, \quad \Gamma_2 = -\frac{1}{2\sqrt{2}} P' e^P \begin{pmatrix} -i\sigma_3 & \sigma_2 \\ \sigma_2 & -i\sigma_3 \end{pmatrix},$$

$$\Gamma_3 = -\frac{1}{2\sqrt{2}} Q' e^Q \begin{pmatrix} i\sigma_2 & \sigma_3 \\ \sigma_3 & i\sigma_2 \end{pmatrix},$$

$$T_{00} = \frac{i\hbar c}{2\sqrt{2}} \{ (u_0^* - u_3^*)(u_0' - u_3') + (u_1^* - u_2^*)(u_1' - u_2') - [(u_0^*)' - (u_3^*)'] \times (u_0 - u_3) - [(u_1^*)' - (u_2^*)'](u_1 - u_2) \};$$

$$T_{11} = \frac{i\hbar c}{2\sqrt{2}} \{ (u_0^* + u_3^*)(u_0' + u_3') + (u_1^* + u_2^*)(u_1' + u_2') - [(u_0^*)' + (u_3^*)'](u_0 + u_3) - [(u_1^*)' + (u_2^*)'](u_1 + u_2) \};$$

$$T_{02} = -\frac{\hbar c}{4} \Big\{ \frac{1}{2} P' e^P [-(u_0^* - u_3^*)(u_0 + u_3) + (u_1^* - u_2^*)(u_1 + u_2) - (u_0^* + u_3^*)(u_0 - u_3) + (u_1^* + u_2^*)(u_1 - u_2)]$$
$$+ e^P [u_0^* u_3' - u_1^* u_2' + u_2^* u_1' - u_3^* u_0' - (u_0^*)' u_3 + (u_1^*)' u_2 - (u_2^*)' u_1 + (u_3^*)' u_0] \Big\};$$

$$T_{03} = \frac{i\hbar c}{4}\left\{\frac{1}{2} Q' e^{Q} [(u_0^* - u_3^*)(u_1 + u_2) - (u_1^* - u_2^*)(u_0 + u_3) + (u_0^* + u_3^*)(u_1 - u_2) - (u_1^* + u_2^*)(u_0 - u_3)] - e^{Q} [u_0^* u_2' - u_1^* u_3' - u_3^* u_1' - (u_0^*)' u_2 + (u_1^*)' u_3 - (u_2^*)' u_0 + (u_3^*)' u_1]\right\};$$

$$T_{23} = \frac{\hbar c}{4\sqrt{2}} (Q' - P') e^{P+Q} \cdot [(u_0^* + u_3^*)(u_1 + u_2) + (u_1^* + u_2^*)(u_0 + u_3)].$$

The system of Einstein – Dirac equations (1), (2) takes on the form

$$\left.\begin{aligned} &-R_{00} \equiv P'' + Q'' + (P')^2 + (Q')^2 = -\frac{8\pi G}{c^4} T_{00}, \\ &T_{0\alpha} \equiv 0, \\ &T_{\alpha\beta} \equiv 0, \\ &2u_0' + u_3' - (P' + Q')(u_0 + u_3) = 0, \\ &2u_1' + u_2' - (P' + Q')(u_1 + u_2) = 0. \end{aligned}\right\} \qquad (7)$$

In the general case system (7) has no solution. However, with an additional assumption as to the form of the wave function $\psi$, system (7) reduces to one unique equation. We assume that the bispinor $\psi$ satisfies the conditions

$$u_0 + u_3 = 0; \quad u_1 + u_2 = 0 \text{ и } u_0 = A + iB; \quad u_1 = C + iD. \qquad (8)$$

From this it is evident that the components $T_{0\alpha}$, $T_{\alpha\beta}$ go to zero, and thus Eq. (7) reduces to a single nonlinear differential equation:

$$P'' + Q'' + (P')^2 + (Q')^2 = \frac{1}{2} \varkappa [(AB' - A'B) + (CD' - C'D)], \qquad (9)$$

where

$$\varkappa = \frac{32\sqrt{\pi}\, G\hbar}{c^3}.$$

If we know the wave function $\psi$ then the gravitational field $g_{ik}$ corresponding to the neutrino flux can be found by integration of the ordinary differential equation (9).

The functions P and Q determine two different states of polarization of the gravitational wave, so that by specifying Q, we find P from Eq. (9), i.e., we define a gravitational wave with completely specified polarization. In fact, assume that Q is known. We take $P = \ln y(x_0)$. Then Eq. (9) reduces to an ordinary second order linear differential equation:

$$y'' = F(x^0) \cdot y, \qquad (10)$$

where

$$F(x^0) = \varkappa [(AB' - A'B) + (CD' - C'D)] - (Q')^2 - Q''.$$

As is known from the theory of differential equations ([7], pp. 75-76), the Cauchy problem for Eq. (10) is uniquely soluble (it is true that solution in quadratures is possible only in special cases) for any continuous function $P = \ln y(x_0)$. Thus we obtain an entire class of solutions of the Einstein—Dirac equations (1), (2).

Metric (5) may always be interpreted as the gravitational radiation of a neutrino flux propagating along the x-axis. In fact, if $g_{ik}$ does not correspond to an empty three-space, then $P'' + Q'' + (P') + (Q')^2 \not\equiv 0$. This means that specifying the functions P and Q, from Eq. (9) we can find the corresponding wave function $\psi$.

As is well known, the neutrino has a well defined longitudinal polarization (the neutrine has a spirality of −1, the antineutrino, + 1). Mathematically, this is expressed by the fact that the wave function $\psi$ of admissable states must satisfy the condition ([2], p. 394):

$$(I - i\gamma_5)\psi = 0,$$

where

$$\gamma_5 = \frac{1}{4!\sqrt{-g}} \varepsilon^{i\kappa lm} \gamma_i \gamma_\kappa \gamma_l \gamma_m.$$

Since Eq. (11) is equivalent to the equalities

$$u_0 + u_2 = 0, \quad u_1 + u_3 = 0, \tag{12}$$

the wave functions considered above, Eq. (8), are admissable, the same condition (8) fixes the sign of the energy, i.e., defines particle or antiparticle.

From Eqs. (8), (12) it follows that A = C, B = D, i.e., Eq. (9) takes on the form:

$$P'' + Q'' + (P')^2 + (Q')^2 = \varkappa(AB' - A'B). \tag{13}$$

## 2. SOLUTIONS OF SPECIAL FORM

Equations (13), (10) are closely related to the Riccati equation, the general solution of which cannot be expressed in quadratures. Consequently, in order to indicate the class of the solutions of Eq. (13), it is necessary to specify beforehand the function Q (i.e., fix the polarization of the gravitational wave) and the wave function $\psi$ or the functions A and B. We will then obtain concrete differential equations, information on which is given in [8]. Nevertheless, one can show a whole class of different solutions of Eq. (13). We will now note some of these.

1. We assume that we are dealing with a single particle (antiparticle), located in a state with specified momentum. Then, as follows from Eqs. (8), (12), the particle's wave function is uniquely defined and has the form

$$\psi(x^0) = \frac{1}{\sqrt{\varkappa}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} e^{ix^0}. \tag{14}$$

We take Q = P and P = ln y ($x^0$). Then, substituting Eq. (14) in Eq. (13), we obtain y" = y, i.e.,

$$y'' = y, \text{ т. е. } P(x^0) = Q(x^0) = \ln(C_1 e^{x^0} + C_2 e^{-x^0}).$$

2. We assume that the wave function $\psi(x^0)$ is known. Further, we take P" = −Q" or

$$Q(x^0) = -P(x^0) + c_1 x^0 + c_2. \tag{15}$$

Then Eq. (13) takes on the form

$$(P')^2 + (-P' + c_1)^2 = \varkappa(AB' - A'B). \tag{16}$$

From Eq. (16) we obtain the solution:

$$P(x^0) = \frac{c_1}{2} x^0 \pm \frac{1}{2} \int \sqrt{2\varkappa(AB' - A'B) - c_1^2}\, dx^0 + c_3 \tag{17}$$

The gravitational wave of Eqs. (15), (17) is a sufficient solution of the problem posed, since for an arbitrary wave function $\psi$ we have found the accompanying gravitational field.

3. Now let the functions A and B be such that AB = AB. This means that the energy-momentum tensor of the neutrino field is identically equal to zero. However, the neutrino flux density is not zero:

$$j^{(\kappa)} = \{4(A^2+B^2),\ -4(A^2+B^2),\ 0,\ 0\},$$

where

$$j^{(\kappa)} = \lambda_i^{(\kappa)} \psi^{+} \gamma^i \psi.$$

In such cases one speaks of a neutrino scent [3, 4]. The gravitational field may then be of the N Petrov type (for example, a space of maximal mobility $T_2$). In contrast, in [3] a field with plane symmetry and a nuetrino scent belonged to the D type, i.e., had no wave structure. Neutrino scents are also possible in a plane space-time. It follows from this that in our treatment the functions P and Q may be constant.

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