Toposes in General Theory of Relativity

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ABSTRACT

We study in this paper different topos-theoretical approaches to the problem of construction of General Theory of Relativity. In general case the resulting space-time theory will be non-classical, different from that of the usual Einstein theory of space-time. This is a new theory of space-time, created in a purely logical manner. Four possibitities are investigated: axiomatic approach to causal theory of space-time, the smooth toposes as a models of Theory of Relativity, Synthetic Theory of Relativity, and space-time as Grothendieck topos.

1 Introduction

Construction of a causal theory of space-time is one of the most attractive tasks of science in the 20th century. From the viewpoint of mathematics, partially ordered structures should be considered. The latter is commonly understood as a set M with a specified reflexive and transitive binary relation $\preceq$. A primary notion is actually not that of causality but rather that of motion (interaction) of material objects. Causality is brought to the foreground since an observer detects changes of object motion or state. It is this detection that gives rise to the view of a particular significance of causes and effects for a phenomenon under study, along with the conviction that causal connections are non-symmetric. Causality is treated as such a relation in the material world that plays a key role in explaining the topological, metric and all other world structures.

Today we imagine space-time as a world, manifold or *set* of elementary (atomic) events. An elementary event is a phenomenon whose extension in both space and time may be neglected. It is assumed that all phenomena *consist of* elementary events. An event is like a point in Euclidean geometry: it is indivisible, or primary. Such an approach allows us to repeat Euclid's way and to arrive at a geometric theory of space-time.

The manifold of events should represent the material world around us. The matter exists in no way than in motion that manifests itself in bodies' influence upon each other. So events also affect each other. Attempting to follow the process of influence, we simplify the interaction picture, concentrate out attention on changing states and thus distinguish causes and effects. So the world appears before our eyes as a full set of most diverse cause-and-effect connections among events.

Today we interpreted the world of events as a *set*. That means that the mathematical modelling of the physical space-time was based on *theory of sets*. This theory has been in the 20th century not only the language used by the mathematicians to formulate and realize their ideas, but also in essence their ideology. This ideology dictates to us the necessity of using of the Cantor's theory of sets for the construction of mathematical causal theory of space-time (see, for example, [1, 2, 3, 4, 5, 6, 7].

Evidently Nature is not forced to be confined in the sets-theoretic ideological frames of mathematical abstractions. A transition *beyond* the frames of theory of sets brings new possibilities for describing the real space-time properties [9]. An event was so far treated as an indivisible phenomenon. This is, however, an evident simplification. Time loops, appearing in general relativity, clearly demonstrate the deficiency of such an approach. A theory should admit the possibility of automatically complicating the elementary (atomic) event structure depending on situation. The structure of (causal) interaction of events should herewith accordingly complicate, as well as the space-time topological and metric structures.

Ideally, it would be necessary to have such a formal theory of space-time that would be able acquire a most sudden appearance relevant to a concrete model. This approach may be exemplified by obtaining in [9] from the same set of axioms, only at the expense of model (topos) choice, either the flat Minkowski space-time, or the curved space-time of general relativity. "Topos theory (see ref. [11, 10]) offers an independent (of the set theory) approach to the foundations of mathematics. Topoi are categories with "set-like", objects, "function-like" arrows and "Boolean-like" logic algebras. Handling "sets", and "functions", in a topos may differ from that in classical mathematics (i.e. the topos **Set** of sets): there are non-classical versions of mathematics, each with its non-Boolean version of logic. One possible view on topoi is this: abstract worlds, universes for mathematical discourse, "inhabitants" (researchers) of which may use non-Boolean logics in their reasoning. From this viewpoint the main business of classical physics is to construct models of the objective (absolute) universe with a given "bivalent Boolean" model of the researcher, and choose the most adequate one" [12].

The topos-theoretical approach to theory of space-time has many preferences since for one formal space-time theory can exist physically different models; each topos gives us own physical world.

Application of topos theory in physics is one of the main intentions of the creator of this theory W.Lawvere [13] The idea of using topoi for construction dynamically variable Universe belongs to russian philosopher I.A.Akchurin [8]. The non-smooth topos theories of space-time was given in [9, 15, 16]. The variants of smooth topos theory of space-time are explained in [17, 18, 19, 20, 21]. Application topos theory in quantum theory can be found in [12, 14].

2 Toposes and Foundation of Theory of Relativity

The system of axioms for the Special theory of relativity contains fewer primery notions and relations, is simple, and lead directly to the ultimate goal (see review [3]). In the case of the General relativity it is difficult to introduce a smoothness. This problem was studied by R.I.Pimenov (see in [6] or presentation of result of [6] in [16]).

Does the unified way of axiomatization of these different physical theories exist? Does the unified way of axiomatization of these different physical theories exist? The language of topos theory [11, 10] gives the unified way of axiomatization of the Special and General Relativity, the axioms being the same in both cases. Selecting one or another physical theory amounts to selecting a concrete topos.

In this section we give a topos-theoretic causal theory of space-time. Let $\mathcal{E}$ be an elementary topos with an object of natural numbers, and let R_T be the object of continuous real numbers [22].

An affine morphism $\alpha : R_T \to R_T$ is a finite composition of morphisms of the form

$$1_{R_T}, \quad \otimes \circ (\lambda \times 1_{R_T}) \circ j, \quad \oplus \circ (1_{R_T} \times \mu) \circ j,$$

where $\oplus, \otimes$ are the operations of addition and multiplication in R_T respectively, λ, μ are arbitrary elements in R_T, and $j : R_T \simeq 1 \times R_T$ is an isomorphism. Let Γ be the set of all affine morphisms from R_T to R_T.

An *affine object* in $\mathcal{E}$ is an object a together with two sets of morphisms:

$$\Phi \subset \mathrm{Hom}_{\mathcal{E}}(R_T, a), \quad \Psi \subset \mathrm{Hom}_{\mathcal{E}}(a, R_T)$$

such that the following conditions hold:

1) For any $\phi \in \Phi, \psi \in \Psi$ there is $\psi \circ \phi \in \Gamma$.

2) If $f \in \mathrm{Hom}_{\mathcal{E}}(R_T, a) \setminus \Phi$ then there exists $\psi \in \Psi$ such that $\psi \circ f \notin \Gamma$.

3) If $f \in \mathrm{Hom}_{\mathcal{E}}(a, R_T) \setminus \Psi$ then there exists $\phi \in \Phi$ such that $f \circ \phi \notin \Gamma$.

4) For any monomorphisms $f : \Omega \mapsto a, g : \Omega \mapsto R_T$ there exists $\phi \in \Phi$ such that $\phi \circ g = f$.

5) For any monomorphisms $f : \Omega \mapsto a, g : \Omega \mapsto R_T$ there exists $\psi \in \Psi$ such that $\psi \circ f = g$.

Here Ω is the subobject classifier in $\mathcal{E}$.

An affine object in category **Set** is the set equipped with an affine structure [23]. In the topos $\mathbf{Bn}(M)$ and in the spatial topos $\mathbf{Top}(M)$ (see notations in [11]), an affine object is a fiber bundle with base M and affines space as fibers.

A categorical description of the Relativity means the introduction of the Lorentz structure either in an affine space or in a fiber bundle with affine spaces as fibers. This can be done by defining in the affine space a family of equal and parallel elliptic cones or a relativistic elliptic conal order [24].

Below we shall use the notations from [11].

Let a be an affine object in the topos $\mathcal{E}$.

Definition 2.1. *An order* in a is an object P together with a collection of subobjects $p_x : P \mapsto a$, where $x : 1 \to a$ is an arbitrary element, such that:

1) $x \in p_x$.

2) If $y \in p_x$, then $z \in p_y$ implies $z \in p_x$.

The order $\langle P, \{p_x\}\rangle$ is denoted as $\mathcal{O}$.

A morphism $f : a \to a$ is called *affine*, if $\psi \circ f \circ \phi \in \Gamma$ for any $\phi \in \Phi$ and $\psi \in \Psi$. We denote the set of all affine morphisms by Aff (a).

Let $\mathcal{A} \subset$ Aff (a) consist of all commuting morphisms. An order $\mathcal{O}$ is *invariant with respect to A* if for any p_x, p_y there exists $g_{xy} \in \mathcal{A}$ such that $g_{xy} \circ p_x = p_y$.

A morphism $f : a \to a$ *preserves* an order $\mathcal{O}$, if for each p_x there exists p_y such that $f \circ p_x = p_y$. The collection of all morphisms preserving an order $\mathcal{O}$ that is invariant with respect to $\mathcal{A}$ is denoted by $Aut(\mathcal{O})$.

A *ray* is a morphism

$$\lambda : R_+ \mapsto R_T \xrightarrow{\varphi} a,$$

where $\phi \in \Phi_0 \subset \Phi$, and for any $\phi \in \Phi_0$ there is no $x : 1 \to a$ such that $\phi = x \circ !$. Here $! : R_T \to 1$ and R_+ is the subobject of object R_T consisting of those t for which $0 \leq t$ (see definition of order in R_T in [22]).

An order $\mathcal{O}$ is called *conic* if 1) for every $y \in p_x$ there exists a ray $\lambda \subset p_x$ such that $x, y \in \lambda$, and 2) x is the origin of λ, i.e. if μ is a ray and $y \in \mu \subset \lambda$, $\mu \neq \lambda$, then $x \notin \mu$.

An order $\mathcal{O}$ has *the acute vertex* or *pointed* one if for each p_x there does not exist $\phi_x \in \Phi_0$ such that $\phi_x \subset p_x$. An order $\mathcal{O}$ is *complete*, if for any element $z : 1 \to a$ and p_x there exist different elements $u_x, v_x : 1 \to a$ and $\phi \in \Phi_0$ such that $z, u_x, v_x \in \phi$ and $u_x, v_x \in p_x$.

An element $u \in p_x$ is called *extreme* if there exists $\phi \in \Phi_0$ for which $u \in \phi$, but $y \notin \phi$ for all $y \in p_x, y \neq u$.

A conic order $\mathcal{O}$ is said to be *strict* if, for each nonextreme element $u \in p_x$, and $v \in p_x, v \neq u$, and each ray λ with origin u such that $v \in \lambda$, there exists an extreme element $w \in \lambda$, and $w \in p_x$.

Definition 2.2. An affine object a with an order $\mathcal{O}$, which is complete, strict, conic, has an acute vertex, and is invariant with respect to $\mathcal{A}$ is said to be *Lorentz* if for each $x : 1 \to a$ and each extreme elements $u, v \in p_x$, where $u, v \neq x$ there exists a $f \in Aut(\mathcal{O})$ such that $f \circ u = v, f \circ x = x$.

Theorem 2.1. *A Lorentz object in the topos* **Set** *is an affine space admitting* *a pseudo-Euclidean structure defined by a quadratic form*

$$x_0^2 - \sum_{i=1}^{n} x_i^2,$$

where n is finite or equal to ∞, and $Aut(\mathcal{O})$ is the Poincaré group (see [24]*). A Lorentz object in the topos* $\mathbf{Top}(M)$ *is a fiber bundle over M with fibers equipped with an affine structure and a continuous pseudo-Euclidean structure of finite or infinite dimension.*

It is quite possible to take not only the topoi **Set**, $\mathbf{Bn}(M)$, or $\mathbf{Top}(M)$, but also any others that have an affine object.

The existing categorical determination of the set theory and determination of $\mathbf{Top}(M)$ between elementary topoi gives the possibility to speak about the solution of problem of categorical description of the Theory of Relativity.

Theorem 2.2. *If $\mathcal{E}$ is a well-pointed topos satisfying the axiom of partial transitivity with a Lorentz object a, then $\mathcal{E}$ is a model of set theory Z and a is a model of the Special Relativity. If $\mathcal{E}$ is a topos defined over* **Set** *that has enough points and satisfies the axiom (SG) (see* [10]*) with a Lorentz object a, then $\mathcal{E}$ is a topos* $\mathbf{Top}(M)$ *and a is a model of the General Relativity.*

3 Smooth toposes as a models of Theory of Relativity

There exists a simple way to construct a non-classical Theory of Relativity with intuitionistic logic by using the objects of smooth toposes

$$\mathbf{Set}^{\mathbf{L}^{\mathbf{op}}},\ \mathbf{Sh}(\mathbf{L}),\ \mathcal{G},\ \mathcal{F}$$

and some others [25]. For this instead of four-dimensional arithmetical space $\mathbb{R}^4$, adopted in sets theory, the role of the world of events passes to its "analogue"

$$R^4 = \ell C^\infty(\mathbb{R}^4) \in \mathbf{Set}^{\mathbf{L}^{\mathbf{op}}},$$

where $\mathbf{L}$ is the category of loci, i.e. the opposite category of finitely genereted C^∞-smooth rings. Every its object ℓA is a C^∞-smooth ring which has the form $\ell A = \ell C^\infty(\mathbb{R}^n)/I$, where $I \subset C^\infty(\mathbb{R}^n)$ is an ideal and symbol ℓ is the label of opposite object. The category $\mathbf{L}$ contains the usual category $\mathbf{M}$ of C^∞-manifolds:

$$\mathbf{M} \subset \mathbf{L} \subset \mathbf{Sh}(\mathbf{L}) \subset \mathbf{Set}^{\mathbf{L}^{\mathbf{op}}}$$

A morphism or arrow in **L** requires more the explicit description:

If $B = C^\infty(\mathbb{R}^n)/J,\ \ A = C^\infty(\mathbb{R}^m)/I$, a morphism $\ell B \to \ell A$ is an equivalence class of smooth function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ with property that $f \in I \Rightarrow f \circ \phi \in J$, while ϕ is equivalent to ϕ' if componentwise, $\phi_i - \phi'_i \in J$ $(i = 1, ..., m)$.

Our viewpoint will be to regard $\mathbf{Set}^{\mathbf{L}^{\mathbf{op}}}$ as a generalized set-theoretic universe, where – intuitively – every set is a smooth space (and the old Cantor sets are embeded as discrete spaces).

An event x as *an element in the locus stage* $\ell A = \ell C^\infty(\mathbb{R}^n)/I \in \mathbb{L}$ of the space-time R^4 is the class of C^∞-smooth vector functions $(X^0(u), X^1(u), X^2(u),\ X^3(u)) : \mathbb{R}^n \to \mathbb{R}^4$, where each function $X^i(u)$ is taken by *mod* I, I is a certain ideal of C^∞-smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$. The argument $u \in \mathbb{R}^n$ is some "hidden" parameter corresponding to the stage ℓA. Hence it follows that at the stage of real numbers $R = \ell C^\infty(\mathbb{R})$ of the topos under consideration an event x is described by just a C^∞-smooth vector function $(X^0(u), X^1(u), X^2(u), X^3(u)), u \in \mathbb{R}$. At the stage of $R^2 = \ell C^\infty(\mathbb{R}^2)$ an event x is 2-dimensional surface, i.e. a *string*. The classical four numbers $(x^0,\ x^1,\ x^2,\ x^3)$, the coordinates of the event x, are obtained at the stage $\mathbf{1} = \ell C^\infty(\mathbb{R}^0) = \ell C^\infty(\mathbb{R})/(t)$ (the ideal (t) allows one to identify functions if their values at 0 coincide), i.e., $x^i = X^i(0), i = 0, 1, 2, 3$.

The space-time transformations $f : R^4 \to R^4$ – are *elements at the stage* ℓA of the functor

$$(R^4)^{R^4} \in \mathbf{Set}^{\mathbf{L}^{\mathbf{op}}},$$

consisting of the classes of C^∞-smooth vector functions $(F^0(u, x), F^1(u, x), F^2(u, x),\ F^3(u, x)) : \mathbb{R}^n \times \mathbb{R}^4 \to \mathbb{R}^4$, where each function $F^i(u, x)$ is taken by *mod* of the ideal $\pi^*(I) = (\phi \circ \pi \mid \phi \in I, \pi : \mathbb{R}^{n+4} \to \mathbb{R}^n$ – projection). At the stage **1** these are ordinary transformations without a "hidden" multidimensional parameter u, while at the stage R these are smooth transformations with a "hidden" parameter.

The relation of causal ordering on R^4 can be defined with the help of the formula

$$\forall x \in R^4 \exists P_x \subset R^4 (x \in P_x \ \&\ (\forall y \in P_x \Rightarrow P_y \subset P_x) \ \&\ (\forall u \neq v \Rightarrow P_u \neq P_v)),$$

where $F \subset G$ means, that a type F is a subtype of type G and one is interpreted in topos $\mathbf{Set}^{\mathbf{L}^{\mathbf{op}}}$ as subfunctor [25]. Essentially, at stage ℓA a set $P_x(\ell A)$ consist of classes C^∞-smooth vector-functions

$$((p_x^0(u), p_x^1(u), p_x^2(u), p_x^3(u)) : \mathbb{R}^n \to \mathbb{R}^4) mod\ I;$$

moreover a term x is interpreted as an element $P_x(\ell A)$, i.e. class $(X^0(u), X^1(u), X^2(u), X^3(u)) mod\ I$. Since R^4 at every stage can equip the vector structure, i.e. convert in vector space, then naturally to think, that set $P_0(\ell A)$ modulo I is semigroup with respect to addition, where 0 at stage ℓA is class $mod\ I$.

The causal automorphisms $f : R^4 \to R^4$ at stage ℓA are defined by relations of the form

$$F(u, X(u) mod\ I + P_0(\ell A)) mod\ \pi^*(I) = F(u, X(u) mod\ I) mod\ \pi^*(I) + P_0(\ell A)$$

for every C^∞-smooth vector function $X(u)$.

It can hope that under some conditions which semigroup $P_0(\ell A)$ must be satisfied (for example, $P_0(\ell A)$ modulo I is an elliptic cone) the order automorphisms are linear operators, which at the stage **1** coinside with Lorentz transformations. In any case one appears the new circle of mathematical and physical problems concerning semigroup theory solving of which will be useful for given here approach to theory of space-time...

It is clearly possible to build a causal topos theory either by analogy with the content of Section 2, or by the scheme used in Sections 4, 5. However, the resulting space-time theory will be non-classical, different from that of the Minkowski space-time. This is a *new* theory of space-time, created in a purely logical manner. It will reflect the real space-time properties to the same extent as the development of mathematical abstractions accompanies the development of the real world.

4 Synthetic Theory of Relativity

In his work W.Lawvere [13] suggested a new approach to the differential geometry and to the others mathematical disciplines which are connected with physics. It allows to give the definitions of derivatives, tangent vectors and tangent bundles without passages to the limits. This approach is based on an idea of consideration generalized or *variable sets* which are the objects of some cartesian closed category $\mathcal{E}$, in particular, of some elementary topos.

The definition of these categories was given by Lawvere and Tirne. They are possessed of their own inner logic, which in general are intuitionistic. So it is possible to formulate mathematical theories with the help of some common logical language and to give proofs of theorems using the laws of intuitionistic logic.

The synthetic differential geometry (SDG) is the theory which was developed by A.Kock [26] in the context of the Lawrere's ideas. Basis of this theory is the assumption that a geometric line is not a filled of real numbers, but is some nondegenerate commutative ring R of a line type, i.e such that it satisfies the following axiom:

Kock-Lawvere Axiom . *Let* $D = \{x \in R \mid x^2 = 0\}$. *For every* $g : D \to R$ *there exist the unique* $a, b \in R$, *such that for any* $d \in D$ *the equation* $g(d) = a + d \cdot b$ *is valid.*

It follows from this axiom that all functions $f : R \to R$ have the first derivatives.

Further we assume another axioms with respect to R which allow to state that any $f : R \to R$ has all derivatives. The main propositions of classical analysis, for example, the properties of derivatives and the Taylor's formula, are valid for these functions.

In SDG the original theories of tangent bundles, differential forms, connections are constructed. For example, a tangent vector to any object M is a map $t : D \to M$ and so the tangent bundle of M is an object M^D.

The SDG has several models, and particular, so called "well adapted models", which allow to compare the classical differential geometry with a synthetic one. These models lie in such categories (toposes) $\mathcal{E}$ that there exist a functors which inset the category $\mathbf{M}$ of C^∞-manifolds in $\mathcal{E}$ (see [26]). For example, we can take a smooth topos as the model for SDG.

In our paper we shall define the basic metrical notions in a context of SDG, and shall show that it is possible to develop pseudo-Riemannian geometry (see [20]) and to write the Einstein's equations. To this end we shall assume that R satisfies to some properties, which are valid in well adapted models.

We denote as $InvR = \{x \in R \mid \exists \ y \in R \ \ x \cdot y = 1\}$ the object of convertible elements in R and define an apartness relation on a cartesian product R^n as it follows: let $x, y \in R^n$, then $x \# y$ iff $\exists i(x_i - y_i \in InvR)$.

We assume that R is a local formally real Pythagorean Archimedean ring and field of quotients [26].

Having the apartness relation on R^n we can develop the theory of intuitionistic linear algebra so as it was made by Heyting in [28]. In particular, the intuitionistic theory of linear equations is valid. The notion of basis of R-module are also defined.

Having the apparat of linear algebra and using the assumptions with respect to R that were given above we define a scalar product on R^n and show that the main metrical properties of R^n do not differ from classical one. For example, if the determinant of matrix of a scalar product is aparted of zero than the matrix is invertible.

For R-modules U and V we define a tensor product $U \otimes V$ and all operations of tensor algebra in a standard manner. Since R^n is a R-module we can define the notions of covariant and contrvariant tensors on R^n, and by using the properties of derivatives and the Taylor's formula we show the existence of analogy of classic tensor analysis. Having a definition of scalar product we define the operation of raising and lowering of indexes.

The next result is the definition a Riemannian tensor on a formal manifold. *A formal manifold* is a notion of SDG which is represent a classical notion of manifold [26]. It has a number of good and natural properties. It is possible to define a local charts of points which are formal etale subojects of R^n, and to show that tangent T_pM space at each point p has the structure of R-module and is isomorphic to R^n.

Let M be a formal manifold. A map $g : TM \times_M TM \to R$ is called *pseudo-Riemannian metric or structure* on M if the following conditions are satisfied:

1. $v \# 0 \Rightarrow \exists u : g(v,u) \# 0$
   $v = 0 \Rightarrow g(v,v) = 0$
2. $g(v,w) = g(w,v)$
3. $g(u+v,w) = g(u,w) + g(v,w)$
4. $g(\lambda \cdot v, w) = \lambda \cdot g(v,w)$

where $\lambda \in R$, $v, w, u \in TM$ so that $v(0) = w(0) = u(0)$.

A map $g_p : T_pM \times T_pM \to R$ is a scalar product on T_pM if $g_p(u,v) = g(u,v)$ for $u, v \in T_pM$,

For any $u, v \in T_pM$ we have $g_p(u,v) = (g_p)_{ij} \cdot u^i v^j$, where u^i, v^j are coordinates of vectors u, v in the basis $\{\partial_i\}$ and $(g_p)_{ij} = g_p(\partial_i, \partial_j)$. We have that $det\|(g_p)_{ij}\| \# 0$ and matrix $\|(g_p)_{ij}\|$ is invertible.

By using the fact that tangent spaces are R-modules we define the tensor bundles under formal manifold.

In [27] was developed the original theory of connections in a context of SDG. There it was shown that connections on the bundles over a subobject U of R^n were defined by $3n$ coefficients Γ^k_{ij} and that the tensor of curvature (tensor of Riemann-Christoffel) had the classical expression in coordinates.

So we define the connection on formal manifold by means of definition of Γ^k_{ij} in a local chart and than introduce the tensor of curvature by using the its coordinates R^l_{ijk} in a local chart which are expressed in a classical manner.

It is evident that in this case the Riemann-Christoffel's tensor has a standart properties. Further we can define the Ricci's and Einstein's tensors by using the classical operations over tensors.

Now we can write the Einstein's equations of gravitational field if we have a four-dimensional formal manifold with a given pseudo-Riemannian structure.

There exist the models of pseudo-Riemannian structure on a formal manifold in different well adapted models.

So we have the method of construction of models of intuitionistic General Theory of Relativity in cartesian closed categories, and, in particullary, in toposes.

In model $\mathbf{Set}^{\mathbf{L}^{\mathbf{op}}}$ the Einstein's equations at stage $\ell A = \ell C^{\infty}(\mathbb{R}^n)/I$ in vacuum have the following form:

$$R_{ij}(u) - \frac{1}{2} g_{ij}(u) R(u) = 0 mod I, \qquad (*)$$

where the argument $u \in \mathbb{R}^n$ is some "hidden" parameter corresponding to the stage ℓA, i.e. we got the non-denumerably infinite collection of equations. The solution of these equations is serious problem. At stage $\mathbf{1} = \ell C^{\infty}(\mathbb{R}^0) = \ell C^{\infty}(\mathbb{R})/(x)$ the equations (*) coinside with the usual Einstein's equations.

5 Space-time as Grothendieck topos

There is still another possibility of applying topos theory to a mathematical description of space-time. One can attempt to achieve the desired simplicity when axiomatizing relativity theory at the cost of giving up the classical view that space-time is the world of events "placed" in a single "space".

To this end, consider a partially ordered set $< \mathbf{P}, \preceq >$ and contravariant functors from the pre-category $\mathbf{P}$ to the topos $\mathbf{Set}$. This gives rise to the topos $\mathbf{Set}^{\mathbf{P}}$, and it is this topos which is the *new mathematical space-time.*

The value of a functor F on an element x of $\mathbf{P}$ is the set $F(x)$. The set $\mathbf{P}$ is interpreted as the collection of all possible situations of obtaining information about past. It has a temporal partial order. The set $F(x)$ is the (causal) past cone consisting of the events that are observed in situation x. The functor F can be interpreted as a time flow. The topos $\mathbf{Set}^{\mathbf{P}}$ consists of all possible time flows. It is not hard to see that a classical Lorentz transformation corresponds to a natural isomorphism of functors, i.e. time flows. In fact, consider two different time flows F and G, and let $x, y, \quad x \preceq y$ are two time situations. Then we have the following diagram:

$$
\begin{array}{ccc}
x & \preceq & y \\
& & \\
F(x) & \stackrel{F(\preceq)}{\longrightarrow} & F(y) \\
\downarrow \tau_x & & \downarrow \tau_y \\
G(x) & \stackrel{G(\preceq)}{\longrightarrow} & G(y)
\end{array}
$$

where τ is a natural isomorphism of functors F, G. Let all $F(x), G(x),\ x \in \mathbf{P}$ are "placed" in a single space $\mathbb{R}^4$. Assume that $F(x)$ is an elliptic cones with vertex x; $F(x)$ is equal and parallel to $F(y)$, $G(x)$ is equal and parallel to $G(y)$ and $\tau_x = \tau : \mathbb{R}^4 \to \mathbb{R}^4$ for all $x \in \mathbf{P}$ is a mapping such that

$$\tau(F(x)) = G(x),$$

$$G(x) = F(\tau(x)) \qquad (**).$$

As it follows from [2, 3] the mapping τ is a classical Lorentz transformation. But if the relation (**) is not valid, for example, $G(x) = \phi_x(F(x))$, where ϕ_x is a rotation on constant angle with respect to point x, $G(x) \neq F(x)$, i.e. we have the flow of time (life) there where we see the present (death), then the theory of sets is useless.

Thus, the space-time $\mathbf{Set}^{\mathbf{P}}$, which may be described as a Grothendieck topos [11], can no longer be "placed" in a single "space".

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